Edge-Disjoint Tree Realization of Tree Degree Matrices that avoid routine induction

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Abstract

Identifying whether a degree matrix has an edge-disjoint realization is an NP-hard problem. In comparison, identifying whether a tree degree matrix has an edge-disjoint realization is easier, but the task is still challenging. In 1975, a sufficient condition for the tree degree matrices with three rows has been found, but the condition has not been improved since. This paper contains an essential part of the proof which improves the sufficient condition.

0 Introduction

Let $V = \{v_1, v_2, \cdots, v_n\}$ be a set of vertices.

Consider the corresponding tree degree matrix

$$D = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} d_{1,1} & d_{1,2} & \cdots & d_{1,n} \\ d_{2,1} & d_{2,2} & \cdots & d_{2,n} \\ d_{3,1} & d_{3,2} & \cdots & d_{3,n} \end{bmatrix},$$

where $d_{i,j}$ represents the degree of the vertex v_j in the i-th tree. Also, let

$$S = \begin{bmatrix} d_{1,1}+d_{2,1}+d_{3,1} & d_{1,2}+d_{2,2}+d_{3,2} & \cdots & d_{1,n}+d_{2,n}+d_{3,n} \end{bmatrix}.$$

Since D_1, D_2, D_3 are tree degree sequences, $\sum_{j=1}^{n} d_{i,j} = 2n-2$ for $i = 1, 2, 3$.

In 1975, Dr Sukhamay Kundu published a paper with the following theorem.

Theorem 0.1. If $\min_{j=1}^{n}\{d_{1,j}+d_{2,j}+d_{3,j}\} = 5$, each tree degree sequence is graphical, and the sum of any two tree degree sequences is graphical, then the matrix has an edge-disjoint tree realization.

In other words, when the conditions are satisfied, it is possible to translate the three tree degree sequences D_1, D_2 and D_3 into a graph such that each degree sequence is translated to a tree (has a tree realization), and no two vertices are connected by more than one edge (is edge-disjoint).[1]

To show whether a tree degree sequence is graphical (can be translated into a graph), the following inequality is used.

Theorem 0.2 (Erdős-Gallai, 1960)**.** A sequence of non-negative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ can be represented as the degree sequence of a finite simple graph on n vertices if and only if $d_1 + d_2 + \cdots + d_n$ is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$$

holds for every k in $1 \leq k \leq n$.

The approach Dr Kundu takes in proving his theorem is mathematical induction. The main part of his proof involves showing that if D contains a column of $\begin{bmatrix}2\\2\\1\end{bmatrix}$ (with respect to symmetry of rows) then we can remove this column and subtract 1 from a high-degree entry (that is, an entry that is greater than 1) in the third row to construct D', and if D' has an edge-disjoint tree realization, then we can construct an edge-disjoint tree realization of D from that of D'.

In the research project I conducted with Dr. Istvan Miklos and Yuhao Wan, we were working on matrices with $\min_{j=1}^{n}\{d_{1,j}+d_{2,j}+d_{3,j}\}=4$. Unfortunately, the routine induction Dr. Kundu utilized does not work when the minimum column sum decreases from 5 to 4, as there are numerous pathological cases.

In this paper, We will investigate one class of those pathological cases, which is the set of matrices that satisfy the following conditions:

- $n \geq 7$
- $\min_{j=1}^{n}\{d_{1,j}+d_{2,j}+d_{3,j}\}=4$
- $\begin{bmatrix}d_{1,1}\\d_{2,1}\\d_{3,1}\end{bmatrix}=\begin{bmatrix}1\\1\\n-3\end{bmatrix}$
- $\begin{bmatrix}d_{1,n}\\d_{2,n}\\d_{3,n}\end{bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix}$
- There is exactly one entry of $n-3$, which is $d_{3,1}$.

The reason why the matrices that satisfy the conditions above are pathological is if we remove the last column, we need to subtract 1 from either $d_{1,1}$ or $d_{2,1}$, but then D' will have an entry of 0, so D' is not a tree degree matrix. More information about routine induction is contained in the next subsection.

Specifically, we will see whether certain tree degree matrices are reducible (able to perform induction by reducing D to D'). Our aim is to show that there are only finitely many irreducible cases, because if this is so, it is possible to check these irreducible cases using a computer program, and all other matrices can be constructed by performing a set of expansions (the process of constructing D from D') from one of these base cases.

Routine Induction

The following is the procedure of reducing D to D' using routine induction when there is a column $\begin{bmatrix}1\\1\\2\end{bmatrix}$ with respect to symmetry of rows.

1. Remove the column $\begin{bmatrix}1\\1\\2\end{bmatrix}$.
2. Choose a high-degree entry from the first row ($d_{1,j}$ such that $d_{1,j}>1$) and subtract the entry by 1.
3. Choose a high-degree entry from the second row and a column different from the j-th column ($d_{2,k}$ such that $d_{2,k}>1$ and $j \neq k$) and subtract the entry by 1.
4. The resulting matrix is D'

Suppose D' has an edge-disjoint tree realization. Introduce the n-th column (hence, n-th vertex, or v_n). v_n connects to v_j via the first tree, and connects to v_k via the second tree. In the third tree, find an edge which connects v_a to v_b ($a, b \neq j, k$) and place v_n in between (so that v_a and v_n are adjacent and v_n and v_b are adjacent, while v_a and v_b are no longer adjacent). We will call the step of placing v_n between v_a and v_b *Archer's Bow* . Once this process is complete, we can verify that D has an edge-disjoint tree realization from the fact that D' has an edge-disjoint tree realization (See **Fig 1** for an example).

$$D = \begin{bmatrix} \mathbf{2} & 2 & 2 & 2 & 2 & 2 & 1 & \mathbf{1} \\ 2 & \mathbf{2} & 2 & 2 & 2 & 1 & 2 & \mathbf{1} \\ 2 & 2 & 2 & 1 & 1 & 2 & 2 & \mathbf{2} \end{bmatrix} \longrightarrow D' = \begin{bmatrix} \mathbf{1} & 2 & 2 & 2 & 2 & 2 & 1 \\ 2 & \mathbf{1} & 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 & 1 & 2 & 2 \end{bmatrix}$$

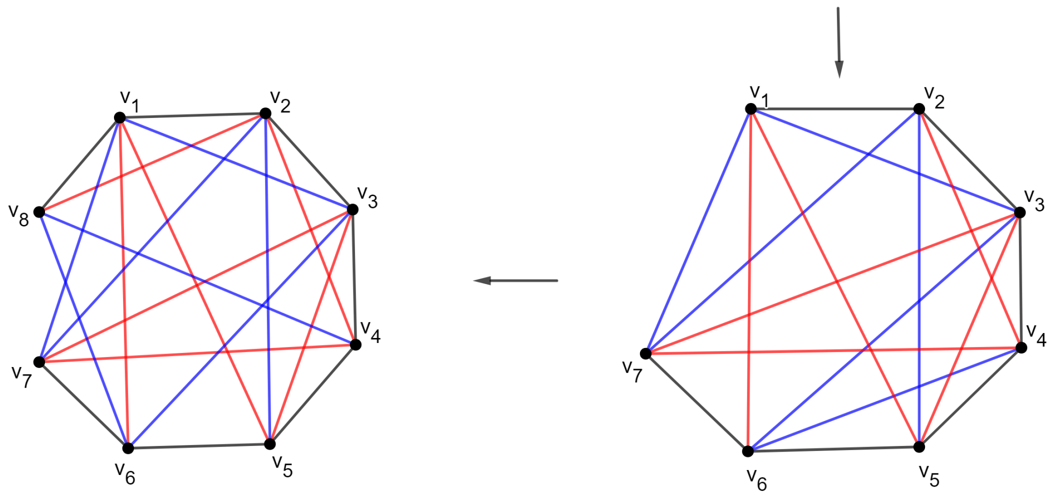

Fig 1. An example of the routine induction

1 Three columns with the sum $n-1$

If D has three columns with the sum $n-1$, D is not reducible. This is because if the minimal sum of each column is 4 (hence, every entry in S is at least 4), no entry of S will be 3, which means D will not have any column of $\begin{bmatrix}1\\1\\1\end{bmatrix}$. This means we can subtract at most two entries of S by 1, so we are still going to have at least one column with the sum $n-1$, which violates the Erdős-Gallai inequality for $k=1$.

Proposition 1.1. If D has three columns with the sum $n-1$, then D has at most 9 columns, so there are finitely many such matrices that cannot be reduced.

Proof. Since the minimum sum of each column is 4, S will look like the following:

$$S = \begin{bmatrix} n-1 & n-1 & n-1 & \geq 4 & \geq 4 & \cdots & \geq 4 \end{bmatrix}$$

Hence, the sum of all the entries in S will be

$$\begin{aligned} 6n-6 &\geq 3(n-1) + 4(n-3) \\ &= 7n-15 \end{aligned}$$

which means, $n \leq 9$. $\square$

2 Two columns with the sum $n-1$

If we can find a column $\begin{bmatrix}1\\2\\1\end{bmatrix}$ or $\begin{bmatrix}2\\1\\1\end{bmatrix}$, we can reduce D to D' using the same method as the routine induction. This will be shown later in the section where we show that if there are at most two columns with the sum $n-1$, then we can always reduce D by removing the column $\begin{bmatrix}1\\2\\1\end{bmatrix}$ or $\begin{bmatrix}2\\1\\1\end{bmatrix}$.

Proposition 2.1. If D has exactly two columns with the sum $n-1$, and no columns of $\begin{bmatrix}1\\2\\1\end{bmatrix}$ or $\begin{bmatrix}2\\1\\1\end{bmatrix}$, then D has at most 8 columns, so there are finitely many such irreducible matrices.

Proof. Note that the third tree degree sequence is

$$D_3 = \begin{bmatrix} n-3 & 1 & 1 & \cdots & 1 & 2 & 2 \end{bmatrix},$$

with respect to symmetry of columns.

Since we cannot find any column of $\begin{bmatrix}1\\2\\1\end{bmatrix}$ or $\begin{bmatrix}2\\1\\1\end{bmatrix}$, each column with the third entry 1 has the sum of at least 5.

Therefore, when a column with the sum of $n-1$ (suppose, without loss of generality, that this is the second column) has $d_{3,2}=1$, D looks like the following:

$$\begin{array}{c} \begin{bmatrix} 1 & d_{1,2} & d_{1,3} & d_{1,4} & \cdots & d_{1,n-1} & 1 \\ 1 & d_{2,2} & d_{2,2} & d_{2,4} & \cdots & d_{2,n-1} & 1 \\ n-3 & 1 & 1 & 1 & \cdots & 2 & 2 \end{bmatrix} \\ S = \begin{bmatrix} n-1 & n-1 & \geq 5 & \geq 5 & \cdots & \geq 4 & 4 \end{bmatrix} \end{array}$$

Then the sum of the all the entries in S is

$$\begin{aligned} 6n-6 &\geq 2(n-1)+5(n-4)+2\cdot 4 \\ &= 7n-14 \end{aligned}$$

Hence, $n \leq 8$.

In the case of $d_{3,2}=2$, D looks like the following:

$$\begin{array}{c} \begin{bmatrix} 1 & d_{1,2} & d_{1,3} & d_{1,4} & \cdots & d_{1,n-1} & 1 \\ 1 & d_{2,2} & d_{2,2} & d_{2,4} & \cdots & d_{2,n-1} & 1 \\ n-3 & 2 & 1 & 1 & \cdots & 1 & 2 \end{bmatrix} \\ S = \begin{bmatrix} n-1 & n-1 & \geq 5 & \geq 5 & \cdots & \geq 5 & 4 \end{bmatrix} \end{array}$$

Then the sum of the all the entries in S is

$$\begin{aligned} 6n - 6 &\geq 2(n-1) + 5(n-3) + 4 \\ &= 7n - 13. \end{aligned}$$

Hence, $n \leq 7$.

After looking at the two cases, we can conclude that $n \leq 8$.

$\square$

Induction on $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Proposition 2.2. If D has at most two columns with the sum $n-1$ and a column of $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (or $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$), then D is reducible, and we can obtain D' by removing the column of $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (or $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$).

Proof. Without loss of generality, suppose D has a column of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ - suppose this is the j-th column.

Then, in D, v_1 connects to v_j via an edge of the third tree.

Now, there must be another high degree (degree greater than 1) in D_1 (suppose this is k-th column) so suppose v_j is connected to v_k via an edge of the first tree in D. If there is another column with the sum $n-1$, then the k-th column must be the column with the sum $n-1$ (because otherwise, D' will have a column with the sum $n-1$, which violates the Erdős-Gallai Inequality of $k=1$). We can always ensure that $d_{1,k} > 1$ when the sum of k-th column is $n-1$, because if $d_{1,k} = 1$, then $d_{2,k} = n-3$, but we assumed earlier that we require D to have at most one entry of $n-3$.

Now, suppose we cannot find an edge in the second tree among $V \setminus \{v_1, v_j, v_k\}$ in D' (so that even if there is a tree realization of D', we cannot perform the *Archer's Bow*and conclude that D also has a tree realization).

Then, the other $n-3$ vertices must be connected to v_k in the second tree, so $d_{2,k} \geq n-3$ (in D'). Since the value of $d_{2,k}$ does not change when we expand D' to D, $d_{2,k} \geq n-3$ in D as well. This is a contradiction, because we required D to have at most one entry of $n-3$ (See **Fig 2.** for the diagram).

Hence, we can find an edge in the second tree among $V \setminus \{v_1, v_j, v_k\}$, so we can perform the routine induction by removing the j-th column when there are at most 2 columns with the sum $n-1$.

Therefore, if D has a column $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (or $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$) and at most two columns with the sum $n-1$, then D is reducible, and D' can be obtained by removing the column $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (or $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$). $\square$

$$D = \begin{bmatrix} 1 & \cdots & d_{1,k} & \cdots & 1 & \cdots & 1 \\ 1 & \cdots & d_{2,k} & \cdots & 2 & \cdots & 1 \\ n-3 & \cdots & d_{3,k} & \cdots & 1 & \cdots & 2 \end{bmatrix}$$

$$D' = \begin{bmatrix} 1 & \cdots & d_{1,k} & \cdots & 1 \\ 1 & \cdots & d_{2,k} & \cdots & 1 \\ n-4 & \cdots & d_{3,k} & \cdots & 2 \end{bmatrix}$$

Fig 2. Reducing D to D' when there is a column of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

3 One column with the sum $n-1$

Proposition 3.1. In D, there exists a column of $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, or $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, with respect to the symmetry of rows.

Proof. It suffices to show that among the columns which have $d_{3,j} = 1$, there exists a column with the sum 4 or 5.

Suppose no columns with $d_{3,j} = 1$ have the sum 4 or 5.

Since $\min_{j=1}^{n}\{d_{1,j} + d_{2,j} + d_{3,j}\} = 4$, all the columns with $d_{3,j} = 1$ have the sum at least 6.

Hence, D will look like the following:

$$\begin{bmatrix} 1 & & & & & & \\ 1 & & & & & & \\ n-3 & 1 & 1 & \cdots & 1 & 2 & 2 \end{bmatrix}$$
$$S = \begin{bmatrix} n-1 & \geq 6 & \geq 6 & \cdots & \geq 6 & \geq 4 & \geq 4 \end{bmatrix}$$

Then, the sum of all the entries in S is

$$\begin{aligned} &6n - 6 \geq (n-1) + 6(n-3) + 2 \cdot 4 = 7n - 11 \\ \Rightarrow\ &n \leq 5 \end{aligned}$$

This is a contradiction, because we required D to have at least 7 columns.

Therefore, we must have a column with $d_{3,j} = 1$ which is $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. $\square$

Proposition 3.2. If D has exactly one column with the sum $n-1$, then there are finitely many such irreducible matrices.

Proof. According to Proposition 3.1, we can always find a column with $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, with respect to the symmetry of D_1 and D_2.

Case 1: D has a column of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Then, D is reducible, according to **Proposition 2.2**.

Case 2: D has no columns of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, but has a column of $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

We will reduce D to D' by removing this column (call this the j-th column).

Assume D' has a realization. We will show that from this assumption, we can conclude that D has a realization.

Connect v_j with v_1 via the third tree. (This means $d_{3,1}$ will become $n-4$ after the reduction.)

Now, let's look at the second row.

The entries in the second row will be $D_2 = \begin{bmatrix} 1 & d_{2,2} & d_{2,3} & \cdots & d_{2,j-1} & 2 & d_{2,j+1} & \cdots d_{2,n-1} & 1) \end{bmatrix}$.

Suppose we cannot find an edge in the second tree of the realization of D' among $\{v_2, v_3, \cdots v_{j-1}, v_{j+1}, \cdots v_{n-1}, v_n\}$ (hence, cannot perform the *Archer's Bow* from D').

This is a contradiction, because this means all these $n-2$ vertices (and $n-2 \geq 1$) must connect to v_1 via the second tree, while $d_{2,1} = 1$.

Hence, we can find an edge in the second tree among $\{v_2, v_3, \cdots v_{j-1}, v_{j+1}, \cdots v_{n-1}, v_n\}$.

Suppose this edge connects v_a and v_b.

Now, we will try to find an edge in the first tree of the realization of D' which is not incident to v_1, v_a nor v_b.

Case 2a: $d_{3,a} = d_{3,b} = 1$

Note that we cannot have $d_{2,a} = d_{2,b} = 1$, because v_a and v_b are connected by an edge in the second tree, and we cannot connect two vertices with degrees 1 in a tree of $n \geq 7$.

Hence, $d_{2,a} + d_{2,b} \geq 3$.

Suppose we cannot find an edge in the first tree of the realization of D' among $V \setminus \{v_1, v_a, v_b, v_j\}$ (hence, cannot perform the *Archer's Bow* from D').

This means all the edges of the first tree of D' are incident to v_1, v_a or v_b.

Then, $d_{1,a} + d_{1,b} \geq n-4$. (This is because in the first tree, even though $v_i \in V \setminus \{v_1, v_a, v_b, v_j\}$ is connected to v_1, v_i must also be connected one of v_a and v_b, so all $n-4$ vertices must somehow connect to v_a or v_b).

Moreover, $d_{1,a} + d_{1,b} \geq n-3$, because in the first tree, one of the vertices of $V \setminus \{v_1, v_a, v_b, v_j\}$ must connect to both d_a and d_b, as the first tree must be connected. (See **Fig 3.** for diagram)

$$D' = \begin{bmatrix} 1 & \cdots & d_{1,a} & \cdots & d_{1,b} & \cdots & 1 \\ 1 & \cdots & d_{2,a} & \cdots & d_{2,b} & \cdots & 1 \\ n-4 & \cdots & 1 & \cdots & 1 & \cdots & 2 \end{bmatrix}$$

Fig 3. An example of D' which satisfies the descriptions above

Therefore, the sum of all the entries in D will be

$$\begin{aligned} 6n - 6 &\geq (n-1) + d_{1,a} + d_{1,b} + d_{2,a} + d_{2,b} + d_{3,a} + d_{3,b} + 5(n-5) + 2 \cdot 4 \\ &\geq (n-1) + n - 3 + 3 + 2 + 5(n-5) + 8 \\ &= 7n - 16 \end{aligned}$$

This means $n \leq 10$. Hence, we have finitely many such matrices D that are irreducible by removing $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

<u>Case 2b</u>: $d_{3,a} = 1, b = n$

Suppose we cannot find an edge in the first tree of the realization of D' among $V \setminus \{v_1, v_a, v_j, v_n\}$.

This means all the vertices in $V \setminus \{v_1, v_a, v_j, v_n\}$ must connect to v_a in the first tree (as $d_{1,1} = d_{1,n} = 1$, so if any vertex connects to one of v_1 and v_n, or both, it must connect to v_a anyways, to make the graph connected). Hence, $d_{1,a} \geq n - 4$.

Since $d_{2,a} \geq 2$ (because v_a is connected to v_b in the second tree and $d_{2,b} = 1$),

$$d_{1,a} + d_{2,a} + d_{3,a} \geq (n-4) + 2 + 1 = n - 1,$$

which is a contradiction, as we required D to have exactly one column with the sum $n-1$, and the column sum cannot exceed $n-1$.

Hence, in the first tree, we must be able to find an edge among $V \setminus \{v_1, v_a, v_j, v_n\}$, so we can reduce D to D' by removing the j-th column, which is $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

<u>Case 2c</u>: $d_{3,a} = 1$, $d_{3,b} = 2$ and $b \neq n$

Then, the situation is similar to Case 2a, but with $d_{3,b} = 2$ instead of 1.

Hence,

$$\begin{aligned} 6n - 6 &\geq (n-1) + d_{1,a} + d_{1,b} + d_{2,a} + d_{2,b} + d_{3,a} + d_{3,b} + 5(n-4) + 4 \\ &\geq (n-1) + n - 3 + 3 + 3 + 5(n-4) + 4 \\ &= 7n - 14 \end{aligned}$$

which means $n \leq 8$.

Therefore, there are finitely many matrices D which cannot be reduced by removing $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

<u>Case 2d</u>: $d_{3,a} = 2, b = n$

Then, the case is similar to Case 2b, except $d_{3,a} = 2$ instead of 1.

Therefore, $d_{1,a} \geq n-4$, $d_{2,a} \geq 2$ and $d_{3,a} = 2$, so $d_a \geq n$, which is a contradiction.

Hence, if D has a column of $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, then there are only finitely many such matrices that cannot be reduced by removing this column.

<u>Case 3</u>: D has no columns of $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, but has columns of $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

Lemma 3.3. If D satisfies $n \geq 11$ and has no columns of $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ ($\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ equivalently), then D must have at least two columns of $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ and two columns of $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

Proof. Assume that D has at most one column of $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

Now, since $d_{1,1} + d_{2,1} + d_{3,1} = n-1$, the sum of the last $n-1$ entries of S is $5n-5$.

Note that we cannot have more than 2 columns with the sum 4, because that means we will obtain a column of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (or $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$).

Therefore, we can have at most 2 columns with the sum 4, and this means we can have at most 5 entries of S that are not 5; S is going to look either like:

$$\begin{bmatrix} n-1 & 6 & 6 & 5 & 5 & \cdots & 5 & 4 & 4 \end{bmatrix},$$

$$\begin{bmatrix} n-1 & 7 & 5 & 5 & 5 & \cdots & 5 & 4 & 4 \end{bmatrix},$$

or

$$\begin{bmatrix} n-1 & 6 & 5 & 5 & 5 & \cdots & 5 & 5 & 4 \end{bmatrix}.$$

Therefore, S will have at least $n-5$ entries of 5.

Furthermore, if we look at D_3, $d_{3,1} = n-3$ and $d_{3,n} = 2$, which means for all columns which satisfy $d_{1,j} + d_{2,j} + d_{3,j} = 5$, there are at least $n-5$ columns which have $d_{3,j} = 1$.

Then, since we don't have any columns of $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, for all j where $d_{1,j} + d_{2,j} + d_{3,j} = 5$, $d_{1,j} = 1$ or 3.

Since we have at most one column of $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, the sum of the entries of D_1 is

$$
\begin{aligned}
&2n-2 \geq (n-6)\cdot 3 + 6\cdot 1 = 3n-12 \\
&\Rightarrow n \leq 10
\end{aligned}
$$

and this is a contradiction to the original assumption.

Using a similar argument, we can conclude that if we have at most one column of $\begin{bmatrix}3\\1\\1\end{bmatrix}$, we get a contradiction.

Hence, if D satisfies $n \geq 11$ and has no columns of $\begin{bmatrix}2\\1\\1\end{bmatrix}$ nor $\begin{bmatrix}2\\2\\1\end{bmatrix}$, then D must have at least two columns of $\begin{bmatrix}1\\3\\1\end{bmatrix}$ and two columns of $\begin{bmatrix}3\\1\\1\end{bmatrix}$.

□

For the matrices with $n \leq 10$, there are finitely many matrices which have less than two columns of $\begin{bmatrix}3\\1\\1\end{bmatrix}$ or less than two columns of $\begin{bmatrix}1\\3\\1\end{bmatrix}$, and hence irreducible using the method which will be outlined below.

Let us look at the four columns: two columns of $\begin{bmatrix}3\\1\\1\end{bmatrix}$ and two columns of $\begin{bmatrix}1\\3\\1\end{bmatrix}$. We will try to reduce D with these four columns into D' with two columns of $\begin{bmatrix}2\\2\\1\end{bmatrix}$.

Suppose v_w and v_z are vertices that correspond to the columns of $\begin{bmatrix}3\\1\\1\end{bmatrix}$ and v_x, v_y are vertices that correspond to the columns of $\begin{bmatrix}1\\3\\1\end{bmatrix}$. Now, consider $(V\setminus\{v_w, v_x, v_y, v_z\})\cup\{v_s, v_t\}$, where v_s, v_t are vertices that correspond to columns of $\begin{bmatrix}2\\2\\1\end{bmatrix}$. Suppose the tree degree matrix which corresponds to $(V\setminus\{v_w, v_x, v_y, v_z\})\cup\{v_s, v_t\}$ is called D' (this means $d_{3,1}$ would have decreased by 2, because D' has 2 less columns than D, and we still need $d_{3,1}$ to be 3 less than the number of columns).

Assume D' has a realization. Suppose T_1 is a subtree that contains a path in the first tree from v_s to v_t, T_2 is a subtree of the first tree that is connected to v_s which is not T_1, and T_3 is a subtree of the first tree that is connected to v_t which is not T_1.

Also, Suppose T_4 is a subtree of the second tree that contains a path from v_s to v_t, T_5 is a subtree of the second tree that is connected to v_s which is not T_4, and T_6 is a subtree of the second tree that is connected to v_t which is not T_4. Then, we can construct the realization as below:

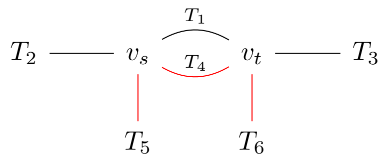

Fig 4. An outline of a realization of D'

Then, we can expand D' to D by using the construction below:

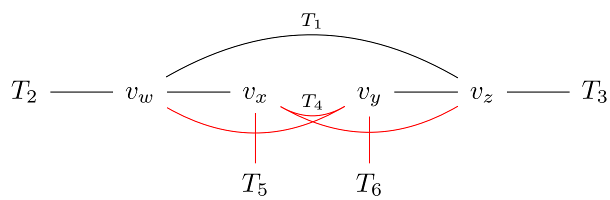

Fig 5. An outline of a realization of D

For the third tree in D', suppose v_s was connected to v_a and v_t was connected to v_b.

Then, in D, connect v_w with v_a, v_y with v_b and connect v_x and v_z with v_1 (so $d_{3,1}$ will increase by 2).

Hence, if D' has a realization, then D also has a realization, so we can reduce D by converting four columns into two columns of $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. $\square$

As an example, consider the following matrix:

$$D = \begin{bmatrix} 1 & 2 & 3 & 3 & 1 & 3 & 1 & 3 & 1 & 1 & 1 \\ 1 & 3 & 2 & 1 & 3 & 1 & 3 & 1 & 3 & 1 & 1 \\ 8 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \end{bmatrix}$$

The process of reducing D to D', and constructing a realization of D from that of D' is shown below in **Fig 6**.

$$D = \begin{bmatrix} 1 & 2 & 3 & 3 & 1 & \mathbf{3} & \mathbf{1} & \mathbf{1} & \mathbf{3} & 1 & 1 \\ 1 & 3 & 2 & 1 & 3 & \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1} & 1 & 1 \\ 8 & 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 2 & 2 \end{bmatrix} \longrightarrow D' = \begin{bmatrix} 1 & 2 & 3 & 3 & 1 & \mathbf{2} & \mathbf{2} & 1 & 1 \\ 1 & 3 & 2 & 1 & 3 & \mathbf{2} & \mathbf{2} & 1 & 1 \\ 6 & 1 & 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & 2 & 2 \end{bmatrix}$$

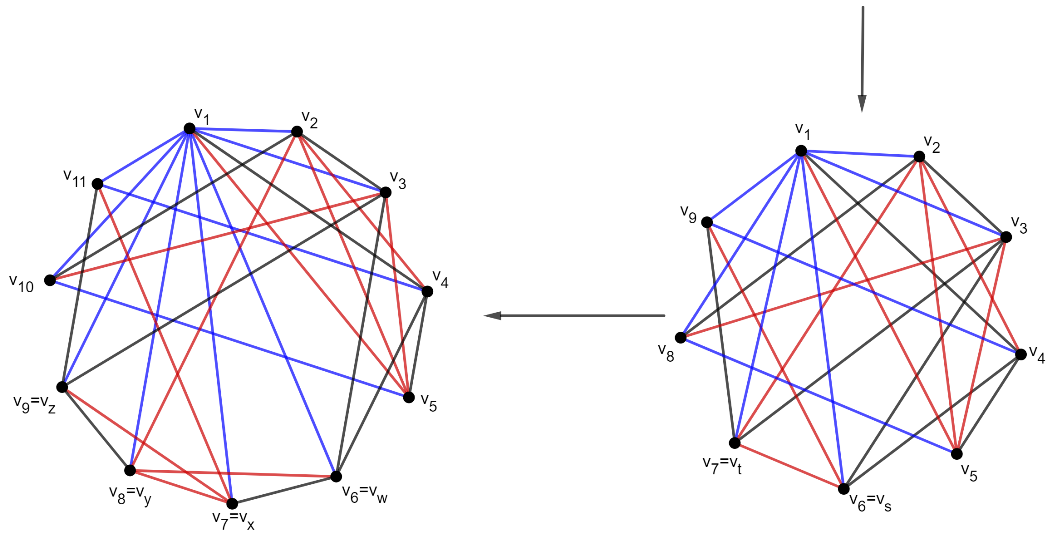

Fig 6. Reducing D to D' in Case 3

4 Conclusion

We showed that if D is a tree degree matrix such that the first column is $\begin{bmatrix} 1 \\ 1 \\ n-3 \end{bmatrix}$, the last column is $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and the minimum sum of each column is 4, then D can always be reduced to D', with the exception of finitely many pathological cases, so that if D' has a tree realization, then so does D.

Other pathological cases have been dealt with by Yuhao Wan, and all the finite exception cases have been checked by a computer program, which could not produce a counter example. Hence, with the help of the results discovered by Dr Istvan Miklos, his computer code, Yuhao Wan, and the result shown in this paper, we conclude the following.

Theorem 4.1. If $\min_{j=1}^{n}\{d_{1,j} + d_{2,j} + d_{3,j}\} = 5$, each tree degree sequence is graphical, and the sum of any two tree degree sequences is graphical, then the matrix has an edge-disjoint tree realization.

This result is expected to be published on the arXiv in 2019.

5 Reference

[1] Kundu, S.: Disjoint Representation of Tree Realizable Sequences. SIAM Journal on Applied Mathematics, 26(1):103–107. (1974)