EECS16A Optional Review Session II (Fall 2015)

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Directions: In groups of 4-5, work on the following exercises. Make sure everyone in the group understands the process before moving on. The rule is that everyone in the entire group must understand a problem before anyone is allowed to move on to the next problem. By helping each other understand, your own understanding will be deepened and strengthened. In some cases, you will discover that you don't really understand it even though you thought you did. This is a good thing because it lets you get help and fix your understanding of these fundamentals right now.

The purpose here in this worksheet is to make sure that you understand the mechanics and fundamentals. The modeling aspects (i.e. "word problems") are purposefully absent here because you have seen many such problems on the homework but it is hard to get them right if you have holes in mechanics and manipulations. So, this worksheet is not meant to be a comprehensive guide to the upcoming midterm or future homework problems. It is here to strengthen your foundations.

* *Asterisked problems are adapted from Linear Algebra by Lipschutz, Seymour and Lipson, Marc, Schaum's Outlines, 5th Ed.*

1 Matrix Dimensions

1. **from Schaum's, Page 76, Theorem 3.9:* Consider a system of equations with n unknowns $x_1, x_2, ..., x_n$. You have m equations. You represent your system of equations like $A\vec{x} = \vec{b}$. For there to be a unique solution, rank(A) must equal what?

2. Find the column rank and span of the following vectors/matrices (*Hint: use Gaussian Elimination when necessary*):

   (a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 2 & 14 \\ 2 & 4 & 5 \\ 3 & 6 & 3 \end{bmatrix}$

   (d) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   (e) **from Schaum's, Page 144, Problem 4.42:* $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix}$

   (f) **from Schaum's, Page 144, Problem 4.41:* $\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

3. **from Schaum's, Page 116, Example 4.3c:* Show that the following vectors do not span $\mathbf{R}^3$: $\vec{u_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{u_2} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\vec{u_3} = \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}$

4. **from Schaum's, Page 159, Problem 4.107:* Which of the following matrices have the same row space?
   $A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -4 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 10 \\ 3 & -5 & 1 \end{bmatrix}$

2 Null Spaces

The null space of a matrix $\mathbf{M}$ is the set of all vectors $\vec{v}$ such that $\mathbf{M}\vec{v} = \vec{0}$. Sometimes this space holds only the zero vector $\vec{0}$, but often times it holds more. Find the null space of each of the following matrices.

i $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

ii $\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$

iii $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

iv $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

v $\begin{bmatrix} -2 & 4 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

vi $\begin{bmatrix} -2 & 4 \\ 3 & 6 \\ 0 & 0 \end{bmatrix}$

vii $\begin{bmatrix} 4 & 1 & 0 \\ 0 & 1 & 5 \\ 3 & 0 & 3 \end{bmatrix}$

viii $\begin{bmatrix} -2 & 0 & 5 \\ 1 & 5 & 0 \\ 0 & -2 & -1 \end{bmatrix}$

ix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

x $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 \\ -1 & 3 & 0 & 5 \end{bmatrix}$

What do you notice about including zero vectors in the matrix?

3 Bases

A basis of a vector space is a minimum set of vectors that span the space.

1. Find a basis for the column space of the following matrices. Note that multiple answers can exist. In general, present your answer as a unit vector or with simple integer values.

   i $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   ii $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   iii $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

   iv $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

   v $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

   vi $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

   vii $\begin{bmatrix} 100 \\ 20 \end{bmatrix}$

   viii $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

   ix $\begin{bmatrix} -1 & 2 & 0 \\ 2 & 2 & 1 \\ 3 & 4 & 0 \end{bmatrix}$

   x $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix}$

   xi $\begin{bmatrix} -2 & 0 & 5 \\ 1 & 5 & 0 \\ 0 & -2 & -1 \end{bmatrix}$

   xii $\begin{bmatrix} 1 & 2 & -3 \\ -1 & -2 & 3 \\ 2 & 4 & -6 \end{bmatrix}$

   xiii $\begin{bmatrix} 1 & 2 & -1 \\ -1 & -3 & 2 \\ 2 & 4 & -2 \end{bmatrix}$

   xiv $\begin{bmatrix} 1 & -2 \\ 1 & -2 \\ 1 & -2 \end{bmatrix}$

   xv $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

   xvi $\begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 0 & -2 & 4 \\ 3 & 5 & 1 & 3 \end{bmatrix}$

   xvii $\begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 0 & -2 & 4 \\ 3 & 5 & 1 & 3 \\ -1 & 2 & 1 & 1 \end{bmatrix}$

   xviii $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 \\ -1 & 3 & 0 & 5 \end{bmatrix}$

2. Find a basis for the subspace spanned by each of these sets of vectors.

   (a) $\begin{bmatrix} 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \end{bmatrix}$

   (b) $\begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$

(c) $\begin{bmatrix}6\\2\\3\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\\-2\end{bmatrix}, \begin{bmatrix}5\\3\\1\\-1\end{bmatrix}$

(d) $\begin{bmatrix}1\\2\\1\\2\end{bmatrix}, \begin{bmatrix}2\\3\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

3. Can a subspace and a basis ever contain the exact same number of vectors? (This is a hard question.)

4. We refer to the number of vectors in the basis of a vector space as the **dimension** of that vector space. Given a matrix $\mathbf{M}$, how are the dimensions of its column space and null space related? What are the maximum and minimum dimensions for each? Hint: relate them to the matrix dimensions of $\mathbf{M}$.

4 Writing Vectors in Other Bases

Another way to look at vectors is a linear combination of basis vectors. Consider:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} = 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

What if we wanted to use a different basis though? Consider:

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\2\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\3\end{bmatrix} = \begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

We have represented the same vector using a different linear combination of new basis vectors. How can we calculate these coefficients? Matrix inverses! If we put the basis vectors into a matrix, we can solve for these coefficients.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$
$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}^{-1}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

i Write the vector $\begin{bmatrix}1\\1\end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

ii Write the vector $\begin{bmatrix}1\\1\end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

iii Write the vector $\begin{bmatrix}1\\1\end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$

iv Write the vector $\begin{bmatrix}1\\2\end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix}0\\-1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}$

v Write the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

vi Write the vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$

vii Write the vector $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$

viii We cannot write $\begin{bmatrix}1\\1\end{bmatrix}$ in terms of $\begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$. Why? What does this tell you about the span of these basis vectors?

5 Determinants

1. Warmup: find the determinant of the following matrices:

   (a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix}$

2. Find the determinant of the following matrix $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

3. These conceptual questions relate to finding the determinant via Gaussian Elimination:

   (a) What happens to the determinant when you scale a row of a matrix? Why?

   (b) What happens to the determinant when you interchange or swap two rows of a matrix? Why? (This is a harder question.)

   (c) What happens to the determinant when you add a multiple of a row to another? Why?

   Note: Try to explain your answers using geometry or a linear algebra proof

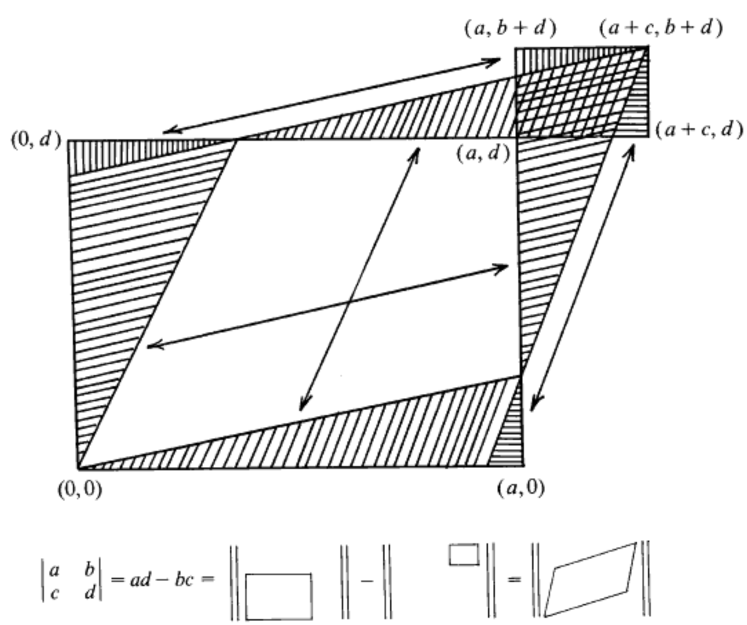
4. Find the determinant of $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 4 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ using row operations.*

5. Find the area of the parallelogram constructed by vectors AB and AC, with $A(1,2), B(-3,4)$ and $C(2,4)$.

6. Calculate the area defined by the same vectors above, but with vectors AC and AB (a different order. How is this area different from your answer from 5?

7. If you still doubt that you can find the area of a parallelogram by calculating the determinant of the matrix comprised of vectors, here's a nice diagram that shows what's going on. Try to explain this diagram (using elementary geometry).
   Source: *Mathematics Magazine, March 1985 by Solomon W. Golomb*

Proof without words:
A 2 × 2 determinant is the area of a parallelogram



6 Find eigenspaces by finding null spaces

First find all the eigenvalues using determinants. Then find the eigenspace for every eigenvalue by finding the appropriate null spaces for the following matrices:

i) $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

ii) $\begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$

iii) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

iv) $\begin{bmatrix} 2 & 4 & 3 \\ 3 & 0 & 2 \\ 4 & -4 & 1 \end{bmatrix}$

v) $\begin{bmatrix} -2 & \frac{-9}{2} & 12 \\ -4 & -3 & 12 \\ -3 & \frac{-7}{2} & 11 \end{bmatrix}$

vi) $\begin{bmatrix} -14 & \frac{-39}{2} & 54 \\ -16 & -15 & 48 \\ -13 & \frac{-29}{2} & 43 \end{bmatrix}$

Note: this matrix is the matrix from part v) squared.

vii) $\begin{bmatrix} -4 & -9 & 24 \\ -8 & -6 & 24 \\ -6 & -7 & 22 \end{bmatrix}$

Note: this matrix is the matrix from part v) multiplied by 2.

7 Transition Matrix and Steady State

Given the following transition matrix, generate the graph and find the steady state solutions. $M_{i,j}$ is the probability that you move from j to i.

$$\begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.7 & 0.2 & 0.1 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$