Fundamentals of Signal Enhancement and Array Signal Processing
Solution Manual

Lidor Malul 318628005

6 An Exhaustive Class of Linear Filters

6.1

Show that the Wiener filter can be expressed as

$$\mathbf{h}_{\mathrm{W}} = \left(\mathbf{I}_M - \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{\Phi}_{\mathrm{in}}\right)\mathbf{i}_{\mathrm{i}}.$$

Solution:
as we know from (6.35):

$$h_W = {\Phi_y}^{-1}\Phi_x i_i$$

which Φ_y is :

$$\Phi_y = \Phi_x + \Phi_{in} \Rightarrow \Phi_x = \Phi_y - \Phi_{in}$$

place this conclusion in (6.35) :

$$h_W = {\Phi_y}^{-1}(\Phi_y - \Phi_{in})i_i = ({\Phi_y}^{-1}\Phi_y - {\Phi_y}^{-1}\Phi_{in})i_i = (I_M - {\Phi_y}^{-1}\Phi_{in})i_i$$

■

6.2

Using Woodbury's identity, show that

$$\mathbf{\Phi}_{\mathbf{y}}^{-1} = \mathbf{\Phi}_{\mathrm{in}}^{-1} - \mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}'_{\mathbf{x}}\left(\mathbf{\Lambda}'^{-1}_{\mathbf{x}} + \mathbf{Q}'^{H}_{\mathbf{x}}\mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}'_{\mathbf{x}}\right)^{-1}\mathbf{Q}'^{H}_{\mathbf{x}}\mathbf{\Phi}_{\mathrm{in}}^{-1}.$$

Solution:
we write Φ_x with his eigenvalue decomposition :

$$\Phi_x = {Q_x}'{\Lambda_x}'{Q_x}'^H$$

now we can express ${\Phi_y}^{-1}$ as:

$${\Phi_y}^{-1} = (\Phi_{in} + \Phi_x)^{-1} = (\Phi_{in} + {Q_x}'{\Lambda_x}'{Q_x}'^H)^{-1}$$

woodbury identity determines that:
if:
Φ_{in} a $M \times M$ reversible matrix
${\Lambda_x}'$ a $R_x \times R_x$ reversible matrix
${Q_x}'$ a $M \times R_x$ matrix
${Q_x}'^H$ a $R_x \times m$ matrix
so:

$$(\Phi_{in} + {Q_x}'{\Lambda_x}'{Q_x}'^H)^{-1} = {\Phi_{in}}^{-1} - {\Phi_{in}}^{-1}{Q_x}'({\Lambda_x}'^{-1} + {Q_x}'^H{\Phi_{in}}^{-1}{Q_x}')^{-1}{Q_x}'^H{\Phi_{in}}^{-1}$$

$$\rightarrow {\Phi_y}^{-1} = {\Phi_{in}}^{-1} - {\Phi_{in}}^{-1}{Q_x}'({\Lambda_x}'^{-1} + {Q_x}'^H{\Phi_{in}}^{-1}{Q_x}')^{-1}{Q_x}'^H{\Phi_{in}}^{-1}$$

■

6.4

Show that the MVDR filter is given by

$$\mathbf{h}_{\text{MVDR}} = \mathbf{\Phi}_{\text{in}}^{-1}\mathbf{Q}_{\mathbf{x}}' \left(\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\text{in}}^{-1}\mathbf{Q}_{\mathbf{x}}'\right)^{-1}\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\text{i}}.$$

Solution:
in order to find the MVDR filter we will solve the following minimization:
$\min_h[J_n(h) + J_i(h)]$ subject to $h^H Q_x{}' = i_i Q_x{}'$
using Lagrange multiplier we define the next function:

$$L(h, \lambda) = f(n) - \lambda g(h)$$

where λ is a $1 \times R_x$ vector and :

$$f(h) = J_n(h) + J_i(h) = \Phi_{vo} h^H h + h^H \Phi_v h = h^H \Phi_{in} h$$

$$g(h) = i_i Q_x{}' - h^H Q_x{}'$$

now we will find the minimum of L :

$$\frac{\partial L(h,\lambda)}{\partial h} = 2\Phi_{in} h - Q_x{}' \lambda^T = 0 \rightarrow h = \frac{1}{2}\Phi_{in}{}^{-1} Q_x{}' \lambda^T$$

$$\frac{\partial L(h,\lambda)}{\partial \lambda} = h^H Q_x{}' - i_i{}^T Q_x{}' = 0 \rightarrow h^H Q_x{}' = i_i{}^T Q_x{}' \rightarrow Q_x{}'^H h = Q_x{}'^H i_i$$

$$Q_x{}'^H h = \frac{1}{2} Q_x{}'^H \Phi_{in}{}^{-1} Q_x{}' \lambda^T = Q_x{}'^H i_i \rightarrow \lambda^T = 2(Q_x{}'^H \Phi_{in}{}^{-1} Q_x{}')^{-1} Q_x{}'^H i_i$$

$$\rightarrow h = \frac{1}{2}\Phi_{in}{}^{-1} Q_x{}' \lambda^T = \Phi_{in}{}^{-1} Q_x{}' (Q_x{}'^H \Phi_{in}{}^{-1} Q_x{}')^{-1} Q_x{}'^H i_i$$

■

6.5

Show that the MVDR filter can be expressed as

$$\mathbf{h}_{\text{MVDR}} = \mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{Q}_{\mathbf{x}}' \left(\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathbf{y}}^{-1}\mathbf{Q}_{\mathbf{x}}'\right)^{-1}\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\text{i}}.$$

Solution:
the MVDR filter is given from the minimiazion of $[J_n(h) + J_i(h)]$
since $[J_d(h)]$ equals 0:

$$[J_n(h) + J_i(h)] = [J_n(h) + J_i(h) + J_d(h)] =$$
$$= \phi_{x1} + h^H \Phi_y h - h^H \Phi_x i_i - i_i{}^T \Phi_x h$$

after the derivative by h all the elements reduce/reset exept from $\frac{\partial h^H \Phi_y h}{d\partial}$
so we continue the previous algorithm with:

$$f(x) = h^H \Phi_y h$$

so the result is:

$$h = \Phi_y{}^{-1} Q_x{}' (Q_x{}'^H \Phi_y{}^{-1} Q_x{}')^{-1} Q_x{}'^H i_i$$

■

6.7

Show that the tradeoff filter can be expressed as

$$\mathbf{h}_{\mathrm{T},\mu} = \mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}_{\mathbf{x}}' \left(\mu\mathbf{\Lambda}_{\mathbf{x}}'^{-1} + \mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{Q}_{\mathbf{x}}'\right)^{-1}\mathbf{Q}_{\mathbf{x}}'^{H}\mathbf{i}_{\mathrm{i}}.$$

Solution:
we know that the tradeoff filter is:

$$h_{T,\mu} = [\Phi_x + \mu\Phi_{in}]^{-1}\Phi_x i_i$$

we use the eigenvalue decomposition of Φ_x :

$$\Phi_x = {Q_x}'{\Lambda_x}'{Q_x}'^H$$

so we get:

$$h_{T,\mu} = [\Phi_x + \mu\Phi_{in}]^{-1}\Phi_x i_i = [\mu\Phi_{in} + {Q_x}'{\Lambda_x}'{Q_x}'^H]^{-1}{Q_x}'{\Lambda_x}'{Q_x}'^H i_i$$

we will also use the following statement which we prove later:

$$(A + VCU)^{-1}U = A^{-1}U(C^{-1} + VA^{-1}U)^{-1}C^{-1}$$

which: A a $M \times M$ reversible matrix
C a $R_x \times R_x$ reversible matrix
U a $M \times R_x$ matrix
V a $R_x \times m$ matrix
so we got:

$$h_{T,\mu} = [\mu\Phi_{in} + {Q_x}'{\Lambda_x}'{Q_x}'^H]^{-1}{Q_x}'{\Lambda_x}'{Q_x}'^H i_i = \frac{1}{\mu}{\Phi_{in}}^{-1}{Q_x}'({\Lambda_x}'^{-1} + \frac{1}{\mu}{Q_x}'^H{\Phi_{in}}^{-1}{Q_x}')^{-1}{\Lambda_x}'^{-1}{\Lambda_x}'{Q_x}'^H i_i$$

$$h_{T,\mu} = {\Phi_{in}}^{-1}{Q_x}'(\mu{\Lambda_x}'^{-1} + {Q_x}'^H{\Phi_{in}}^{-1}{Q_x}')^{-1}{Q_x}'^H i_i$$

■

prove for the statement we used:

$$(A + UCV)^{-1}U = (A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1})U =$$

$$= A^{-1}U - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}U = A^{-1}U(C^{-1} + VA^{-1}U)^{-1}[C^{-1} + VA^{-1}U - VA^{-1}U] =$$

$$= A^{-1}U(C^{-1} + VA^{-1}U)^{-1}C^{-1}$$

■

6.8

Show that the LCMV filter is given by

$$\mathbf{h}_{\mathrm{LCMV}} = \mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{C}_{\mathbf{x}\mathbf{v}_1}\left(\mathbf{C}_{\mathbf{x}\mathbf{v}_1}^{H}\mathbf{\Phi}_{\mathrm{in}}^{-1}\mathbf{C}_{\mathbf{x}\mathbf{v}_1}\right)^{-1}\mathbf{i}_{\mathrm{c}}.$$

Solution:
in order to find the LCMV filter we will solve the following minimization:
$\min_h[J_n(h) + J_i(h)]$ subject to $h^H {C_{xv1}}' = i_i$
using Lagrange multiplier we define the next function:

$$L(h, \lambda) = f(n) - \lambda g(h)$$

where λ is a $1 \times R_x$ vector and :

$$f(h) = J_n(h) + J_i(h) = \Phi_{vo}h^H h + h^H\Phi_v h = h^H\Phi_{in}h$$

$$g(h) = i_i - h^H {C_{xv1}}'$$

now we will find the minimum of L :

$$\frac{\partial L(h,\lambda)}{\partial h} = 2\Phi_{in}h - {C_{xv1}}^{'}\lambda^T = 0 \rightarrow h = \frac{1}{2}{\Phi_{in}}^{-1}{C_{xv1}}^{'}\lambda^T$$

$$\frac{\partial L(h,\lambda)}{\partial \lambda} = h^H{C_{xv1}}^{'} - {i_i}^T{C_{xv1}}^{'} = 0 \rightarrow h^H{C_{xv1}}^{'} = {i_i}^T{C_{xv1}}^{'} \rightarrow {C_{xv1}}^{'H}h = {C_{xv1}}^{'H}i_i$$

$${C_{xv1}}^{'H}h = \frac{1}{2}{C_{xv1}}^{'H}{\Phi_{in}}^{-1}{C_{xv1}}^{'}\lambda^T = {C_{xv1}}^{'H}i_i \rightarrow \lambda^T = 2({C_{xv1}}^{'H}{\Phi_{in}}^{-1}{C_{xv1}}^{'})^{-1}{C_{xv1}}^{'H}i_i$$

$$\rightarrow h_{LCMV} = \frac{1}{2}{\Phi_{in}}^{-1}{C_{xv1}}^{'}\lambda^T = {\Phi_{in}}^{-1}{C_{xv1}}^{'}({C_{xv1}}^{'H}{\Phi_{in}}^{-1}{C_{xv1}}^{'})^{-1}{C_{xv1}}^{'H}i_i$$

■

6.10

Show that the LCMV filter can be expressed as

$$\mathbf{h}_{\text{LCMV}} = \mathbf{Q}''_{\mathbf{v}_1}\mathbf{\Phi}'^{-1}_{\text{in}}\mathbf{Q}''^H_{\mathbf{v}_1}\mathbf{Q}'_{\mathbf{x}}\left(\mathbf{Q}'^H_{\mathbf{x}}\mathbf{Q}''_{\mathbf{v}_1}\mathbf{\Phi}'^{-1}_{\text{in}}\mathbf{Q}''^H_{\mathbf{v}_1}\mathbf{Q}'_{\mathbf{x}}\right)^{-1}\mathbf{Q}'^H_{\mathbf{x}}\mathbf{i}_{\text{i}}.$$

Solution:
in order to find the LCMV filter a we will solve the following minimization:
$\min_h[J_n(a) + J_i(a)]$ subject to ${i_i}^T{Q_x}^{'} = a^H{Q_{v1}}^{''H}{Q_x}^{'}$
using Lagrange multiplier we define the next function:

$$L(h,\lambda) = f(n) - \lambda g(h)$$

where λ is a $1 \times R_x$ vector and :

$$f(a) = J_n(a) + J_i(a) = \Phi_{vo}a^Ha + a^H\Phi_v h = h^H\Phi_{in}h$$

$$g(a) = {i_i}^T{Q_x}^{'} - a^H{Q_{v1}}^{''H}{Q_x}^{'}$$

now we will find the minimum of L :

$$\frac{\partial L(a,\lambda)}{\partial a} = 2\Phi_{in}a - {Q_{v1}}^{''H}{Q_x}^{'}\lambda^T = 0 \rightarrow a = \frac{1}{2}{\Phi_{in}}^{-1}{Q_{v1}}^{''H}{Q_x}^{'}\lambda^T$$

$$\frac{\partial L(a,\lambda)}{\partial \lambda} = 0 \rightarrow g(a) = 0 \rightarrow a^H{Q_{v1}}^{''H}{Q_x}^{'} = {i_i}^T{Q_x}^{'} \rightarrow {Q_{v1}}^{''}{Q_x}^{'H}a = {Q_x}^{'H}i_i$$

$${Q_{v1}}^{''}{Q_x}^{'H}a = \frac{1}{2}{Q_{v1}}^{''}{Q_x}^{'H}{\Phi_{in}}^{-1}{Q_{v1}}^{''H}{Q_x}^{'}\lambda^T = {Q_x}^{'H}i_i \rightarrow \lambda^T = 2({Q_{v1}}^{''}{Q_x}^{'H}{\Phi_{in}}^{-1}{Q_{v1}}^{''H}{Q_x}^{'})^{-1}{Q_x}^{'H}i_i$$

$$\rightarrow a_{LMCV} = {\Phi_{in}}^{-1}{Q_{v1}}^{''H}{Q_x}^{'}({Q_{v1}}^{''}{Q_x}^{'H}{\Phi_{in}}^{-1}{Q_{v1}}^{''H}{Q_x}^{'})^{-1}{Q_x}^{'H}i_i$$

■

6.11

Show that the maximum SINR filter with minimum distortion is given by

$$\mathbf{h}_{\text{mSINR}} = \frac{\mathbf{t}_1\mathbf{t}_1^H\mathbf{\Phi}_{\mathbf{x}}\mathbf{i}_{\text{i}}}{\lambda_1} = \mathbf{t}_1\mathbf{t}_1^H\mathbf{\Phi}_{\text{in}}\mathbf{i}_{\text{i}}.$$

Solution:
we know the maximum SINR filter is given by:

$$h_{mSINR} = t_1\varsigma$$

where ς is an arbitrary complex number, determine by solving the following minimization :

$$J_d(h_{mSINR}) = \Phi_{x1} + \lambda_1|\varsigma|^2 - \varsigma^* {t_1}^H \Phi_x i_i - \varsigma {i_i}^T \Phi_x t_1$$
$$\frac{\partial J_d}{\partial \varsigma^*} = 2\lambda_1 \varsigma - {t_1}^H \Phi_x i_i - ({i_i}^T \Phi_x t_1)^H = 0$$
$$2\lambda_1 \varsigma - {t_1}^H \Phi_x i_i - {t_1}^H \Phi_x i_i = 0 \rightarrow \varsigma = \frac{{t_1}^H \Phi_x i_i}{\lambda_1}$$

so the maximum SINR filter is:

$$h_{sSINR} = \frac{t_1 {t_1}^H \Phi_x i_i}{\lambda_1}$$

■

6.13

Show that the output SINR can be expressed as

$$\begin{aligned} \text{oSINR}(\mathbf{a}) &= \frac{\mathbf{a}^H \boldsymbol{\Lambda} \mathbf{a}}{\mathbf{a}^H \mathbf{a}} \\ &= \frac{\sum_{i=1}^{R_x} |a_i|^2 \lambda_i}{\sum_{m=1}^{M} |a_m|^2}. \end{aligned}$$

Solution:
let's remember the definition of oSINR:

$$oSINR = \frac{h^H \Phi_x h}{h^H \Phi_{in} h}$$

where h writed in a basis formed:

$$h = Ta$$

from (6.83) and (6.84):

$$T^H \Phi_x T = \Lambda$$
$$T^H \Phi_{in} T = I_M$$

we use all of that and substituting at the definition of oSINR:

$$\frac{h^H \Phi_x h}{h^H \Phi_{in} h} = \frac{a^H T^H \Phi_x T a}{a^H T^H \Phi_{in} T a} = \frac{a^H \Lambda a}{a^H I_M a}$$
$$\rightarrow oSINR = \frac{a^H \Lambda a}{a^H a}$$

■

6.14

Show that the transformed identity filter, $\mathbf{i_T}$, does not affect the observations, i.e., $z = \mathbf{i_T}^H \mathbf{T}^H \mathbf{y} = y_1$ and $\text{oSINR}(\mathbf{i_T}) = \text{iSINR}$.
Solution:
we know that z is :

$$z = a^H T^H y$$

for $a = i_T$ we get:

$$z = {i_T}^H T^H y = (T^{-1} i_i)^H T^H y = {i_i}^H T^{-1H} T^H y = i_i y$$
$$\rightarrow z = y_1$$

■

6.16

Show that the MSE can be expressed as

$$J(\mathbf{a}) = (\mathbf{a} - \mathbf{i_T})^H \mathbf{\Lambda} (\mathbf{a} - \mathbf{i_T}) + \mathbf{a}^H \mathbf{a}.$$

Solution:
as we know from (6.83):

$$T^H \Phi_x T = \Lambda \rightarrow \Phi_x = T^{H-1} \Lambda T^{-1}$$

$$\phi_{x1} = {i_i}^H \Phi_x i_i \rightarrow \phi_{x1} = {i_i}^H T^{H-1} \Lambda T^{-1} i_i$$

now we will simplify the MSE from section 6.15:

$$J(a) = \phi_{x1} - a^H \Lambda i_T - i_T \Lambda a + a^h (\Lambda + I_M) a$$

as we prove before:

$$\phi_{x1} = {i_i}^H T^{H-1} \Lambda T^{-1} i_i = (T^{-1} i_i)^H \Lambda \left(T^{-1} i_i\right)$$

$$\rightarrow \phi_{x1} = {i_T}^H \Lambda i_T$$

$$\rightarrow J(a) = \phi_{x1} - a^H \Lambda i_T - i_T \Lambda a + a^h (\Lambda + I_M) a = {i_T}^H \Lambda i_T - a^H \Lambda i_T - i_T \Lambda a + a^h \Lambda a + a^h I_M a$$

$$= a^H \Lambda (a - i_T) - {i_T}^H \Lambda (a - i_T) + a^H a = (a^H \Lambda - {i_T}^H \Lambda)(a - i_T) + a^H a =$$

$$= (a^H - {i_T}^H) \Lambda (a - i_T) + a^H a = (a - i_T)^H \Lambda (a - i_T) + a^H a$$

$$J(a) = (a - i_T)^H \Lambda (a - i_T) + a^H a$$

■

6.17

Show that the maximum SINR filter with minimum MSE is given by

$$\mathbf{h}_{\mathrm{mSINR},2} = \frac{\lambda_1}{1 + \lambda_1} \mathbf{t}_1 \mathbf{t}_1^H \mathbf{\Phi}_{\mathrm{in}} \mathbf{i}_{\mathrm{i}}.$$

Solution:
first of all we know from the definition of T :

$$(1). T i_i = t_1$$

$$(2). T^H \Phi_{in} T = I_M$$

$$\rightarrow {i_i}^T = {i_i}^T I_M = {i_i}^T T^H \Phi_{in} T = (T i_i)^H \Phi_{in} T = {t_1}^H \Phi_{in} T$$

$$\rightarrow {i_i}^T T^{-1} = {t_1}^H \Phi_{in} T T^{-1} = {t_1}^H \Phi_{in}$$

as we know about a_{mSINR} and the conclusions we shown before:

$$a_{mSINR} = \frac{\lambda_1}{1 + \lambda_1} i_i {i_i}^T T^{-1} i_i = \frac{\lambda_1}{1 + \lambda_1} i_i {t_1}^H \Phi_{in} i_i$$

$$h_{mSINR} = T a_{mSINR} = \frac{\lambda_1}{1 + \lambda_1} T i_i {t_1}^H \Phi_{in} i_i$$

now we use the identity (1) that we mention earlier:

$$h_{mSINR} = \frac{\lambda_1}{1 + \lambda_1} t_1 {t_1}^H \Phi_{in} i_i$$

■

6.19

Show that the Wiener filter can be expressed as

$$\mathbf{h}_{\mathrm{W}} = \sum_{i=1}^{R_x} \frac{\lambda_i}{1+\lambda_i} \mathbf{t}_i \mathbf{t}_i^H \mathbf{\Phi}_{\mathrm{in}} \mathbf{i}_{\mathrm{i}}.$$

Solution:
first of all we know from the definition of T :

$$(1).Ti_i = t_1$$
$$(2).T^H \Phi_{in} T = I_M$$
$$\to {i_i}^T = {i_i}^T I_M = {i_i}^T T^H \Phi_{in} T = (Ti_i)^H \Phi_{in} T = {t_1}^H \Phi_{in} T$$
$$\to {i_i}^T T^{-1} = {t_1}^H \Phi_{in} T T^{-1} = {t_1}^H \Phi_{in}$$

as we know about a_W and the conclusions we shown before:

$$a_W = \sum_{i=1}^{R_X} \frac{\lambda_i}{1+\lambda_i} i_i {i_i}^T T^{-1} i_i = \sum_{i=1}^{R_X} \frac{\lambda_i}{1+\lambda_i} i_i {t_1}^H \Phi_{in} i_i$$

$$h_w = T a_W = T \sum_{i=1}^{R_X} \frac{\lambda_i}{1+\lambda_i} i_i {t_1}^H \Phi_{in} i_i = \sum_{i=1}^{R_X} \frac{\lambda_i}{1+\lambda_i} T i_i {t_1}^H \Phi_{in} i_i$$

now we use the identity (1) that we mention earlier:

$$h_w = \sum_{i=1}^{R_X} \frac{\lambda_i}{1+\lambda_i} t_1 {t_1}^H \Phi_{in} i_i$$

■

6.20

Show that with the Wiener filter $\mathbf{h}_{\mathrm{W}}$, the MMSE is given by

$$J(\mathbf{h}_{\mathrm{W}}) = \mathbf{i}_{\mathbf{T}}^H \mathbf{\Lambda} \mathbf{i}_{\mathbf{T}} - \sum_{i=1}^{R_x} \frac{\lambda_i^2}{1+\lambda_i} \left| \mathbf{i}_{\mathbf{T}}^H \mathbf{i}_i \right|^2$$
$$= \sum_{i=1}^{R_x} \frac{\lambda_i}{1+\lambda_i} \left| \mathbf{i}_{\mathbf{T}}^H \mathbf{i}_i \right|^2.$$

Solution:
As was shown before:

$$J(h_W) = J(a_W)$$

we also know :

$$a_W = \sum_{i=1}^{R_X} \frac{\lambda_i}{1+\lambda_i} i_i {i_i}^T i_T$$

$${a_W}^H = \sum_{i=1}^{R_X} \frac{\lambda_i}{1+\lambda_i} {i_T}^H i_i {i_i}^T$$

so we will calculate $J(a_W)$:

$$J(a_W) = (a_w - i_T)^H \Lambda (a_w - i_T) + {a_W}^H a_W =$$
$$= {i_T}^H \Lambda i_T + {a_W}^H \Lambda a_W - {i_T}^H \Lambda a_W - {a_W}^H \Lambda i_T + {a_W}^H a_W$$

now let's calculate each part separately:

$$a_W{}^H a_W = \sum_{i=1}^{R_X} (\frac{\lambda_i}{1+\lambda_i})^2 i_T{}^H i_i i_i{}^T i_i i_i{}^T i_T = \sum_{i=1}^{R_X} (\frac{\lambda_i}{1+\lambda_i})^2 \left|i_T{}^H i_i\right|^2$$

$$i_T{}^H \Lambda a_W = \sum_{i=1}^{R_X} \frac{\lambda_i}{1+\lambda_i} i_T{}^H \Lambda i_i i_i{}^T i_T = \sum_{i=1}^{R_X} \frac{\lambda_i{}^2}{1+\lambda_i} i_T{}^H i_i i_i{}^T i_i i_i{}^T i_T = \sum_{i=1}^{R_X} \frac{\lambda_i{}^2}{1+\lambda_i} \left|i_T{}^H i_i\right|^2$$

$$a_W{}^H \Lambda i_T = \sum_{i=1}^{R_X} \frac{\lambda_i}{1+\lambda_i} i_T{}^H i_i i_i{}^T \Lambda i_T = \sum_{i=1}^{R_X} \frac{\lambda_i{}^2}{1+\lambda_i} i_T{}^H i_i i_i{}^T i_i i_i{}^T i_T = \sum_{i=1}^{R_X} \frac{\lambda_i{}^2}{1+\lambda_i} \left|i_T{}^H i_i\right|^2$$

$$a_W{}^H \Lambda a_W = \sum_{i=1}^{R_X} (\frac{\lambda_i}{1+\lambda_i})^2 \lambda_i i_T{}^H i_i i_i{}^T i_i i_i{}^T i_T = \sum_{i=1}^{R_X} (\frac{\lambda_i}{1+\lambda_i})^2 \lambda_i \left|i_T{}^H i_i\right|^2$$

We will put everything into our expression:

$$J(a_W) = i_T{}^H \Lambda i_T + \sum_{i=1}^{R_X} \left(\left(\frac{\lambda_i}{1+\lambda_i}\right)^2 + \lambda_i \left(\frac{\lambda_i}{1+\lambda_i}\right)^2 - 2\frac{\lambda_i{}^2}{1+\lambda_i} \right) \left|i_T{}^H i_i\right|^2 = i_T{}^H \Lambda i_T + \sum_{i=1}^{R_X} \left(\left(\frac{\lambda_i}{1+\lambda_i}\right)^2 (1+\lambda_i) - 2(1+\lambda_i)\left(\frac{\lambda_i}{1+\lambda_i}\right)^2 \right) |$$

$$= i_T{}^H \Lambda i_T + \sum_{i=1}^{R_X} \left(\frac{\lambda_i{}^2}{1+\lambda_i} - 2\frac{\lambda_i{}^2}{1+\lambda_i} \right) \left|i_T{}^H i_i\right|^2 = i_T{}^H \Lambda i_T - \sum_{i=1}^{R_X} \left(\frac{\lambda_i{}^2}{1+\lambda_i} \right) \left|i_T{}^H i_i\right|^2$$

finally let's simplify the expression:

$$J(h_W) = i_T{}^H \Lambda i_T - \sum_{i=1}^{R_X} \left(\frac{\lambda_i{}^2}{1+\lambda_i} \right) \left|i_T{}^H i_i\right|^2 = \sum_{i=1}^{R_X} \lambda_i \left|i_T{}^H i_i\right|^2 - \sum_{i=1}^{R_X} \left(\frac{\lambda_i{}^2}{1+\lambda_i} \right) \left|i_T{}^H i_i\right|^2 = \sum_{i=1}^{R_X} \left(\lambda_i - \frac{\lambda_i{}^2}{1+\lambda_i} \right) \left|i_T{}^H i_i\right|^2 = \sum_{i=1}^{R_X} \frac{\lambda_i}{1+\lambda_i} \left|i_T{}^H i_i\right|^2$$

.

■

6.22

Show that the class of filters $\mathbf{a}_Q$ compromises in between large values of the output SINR and small values of the MSE, i.e.,

$$(a) iSNR \leq oISNR(a_{R_X}) \leq oISNR(a_{R_X-1}) \leq \cdots \leq oISNR(a_1) = \lambda_1$$

$$(b) J(a_{R_X}) \leq J(a_{R_X-1}) \leq \cdots \leq J(a_1)$$

Solution:
first of all we will use the following property:
Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M \geq 0$

$$\frac{\sum_{i=1}^{M} |a_i|^2 \lambda_i}{\sum_{i=1}^{M} |a_i|^2} \leq \frac{\sum_{i=1}^{M-1} |a_i|^2 \lambda_i}{\sum_{i=1}^{M-1} |a_i|^2} \leq \cdots \leq \frac{\sum_{i=1}^{2} |a_i|^2 \lambda_i}{\sum_{i=1}^{2} |a_i|^2} \leq \lambda_1$$

now we can define a class of filters that have the following form:

$$a_Q = \sum_{q=1}^{Q} \frac{\lambda_q}{1+\lambda_q} i_q i_q{}^T T^{-1} i_i$$

where $1 \leq Q \leq R_X$ we can easly see:

$$h_1 = h_{mSINR,2}$$

$$h_{R_X} = h_W$$

from the property we shown earlier it is easy to see that:

$$iSNR \leq oSNR(a_{R_X}) \leq oSNR(a_{R_X-1}) \leq \cdots \leq oSNR(a_1) = \lambda_1$$

■

now it is easy to compute the MSE:

$$J(a_Q) = {i_T}^H \Lambda i_T - \sum_{q=1}^{Q} \frac{{\lambda_q}^2}{1+\lambda_q} \left| {i_T}^H i_q \right|^2 = \sum_{q=1}^{Q} \frac{{\lambda_q}^2}{1+\lambda_q} \left| {i_T}^H i_q \right|^2 + \sum_{i=Q+1}^{R_X} \lambda_i \left| {i_T}^H i_q \right|^2$$

so finally we can deduce that:

$$J(a_{R_X}) \leq J(a_{R_X-1}) \leq \cdots \leq J(a_1)$$

■