Inverse Function Integral Theorem

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In this document a theorem shall be introduced dealing with evaluating improper integrals as proper integrals of inverse functions. To start, the theorem is expressed below.

Theorem

For any integrable function $f(x)$ the derivative $f'(x)$ of which is always either positive or negative in the domain $x \in \mathbb{R}$, and that has a limit $\lim_{x\to\infty} f(x) = 0$, its integral can be expressed as

$$\int_0^\infty f(x)\,dx = \int_0^{\lim_{x\to 0} f(x)} f^{-1}(x)\,dx$$

Proof

One may consider an arbitrary function $f(x)$ given by

$$f(x) = y$$

If the variables are swapped as such,

$$f(y) = x$$

y can be said to be the inverse function of x,

$$y = f^{-1}(x)$$

Consider the integrals of a function and its inverse,

$$\int f(x)\,dx = F(x)$$

and

$$\int f^{-1}(x)\,dx = F^{-1}(x)$$

where $F(x)$ and $F^{-1}(x)$ are inverses of each other. The integrals can be defined and done with boundaries of zero to infinity, and thus one has

$$\int_0^\infty f(x)\,dx = \lim_{x\to\infty} F(x) - \lim_{x\to 0} F(x)$$

The function $f(x)$ must have a limit

$$\lim_{x\to\infty} f(x) = 0$$

and therefore its inverse will have the limit

$$\lim_{x\to 0} f^{-1}(x) = \infty$$

From this one can say that

$$f^{-1}\left(f(0)\right) = 0$$

This substitution can be made into the integral of $f(x)$ as such,

$$\int_0^\infty f(x)\,dx = \lim_{x\to f^{-1}(0)} F(x) - \lim_{x\to f^{-1}(f(0))} F(x)$$

Take the inverse of the functions on the left side,

$$\int_0^\infty f(x)\,dx = \lim_{x\to f(0)} F^{-1}(x) - \lim_{x\to f(f^{-1}(0))} F^{-1}(x)$$

$f\left(f^{-1}(0)\right)$ can be simplified as

$$f\left(f^{-1}(0)\right) = \lim_{x\to\infty} f(x) = 0$$

The integral now becomes

$$\int_0^\infty f(x)\,dx = \lim_{x\to f(0)} F^{-1}(x) - \lim_{x\to 0} F^{-1}(x)$$

which can be simplified into the integral of the inverse function,

$$\int_0^\infty f(x)\,dx = \int_0^{\lim_{x\to 0} f(x)} f^{-1}(x)\,dx$$

If $f(x)$ has an output at zero, the equation can be shown as

$$\int_0^\infty f(x)\,dx = \int_0^{f(0)} f^{-1}(x)\,dx$$

Q.E.D.