The limit and convergence of sequences

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Abstract

This document is a short introduction to the idea of convergence and divergence in sequences and shows how to prove a sequence is convergent by using the definition of convergence. This document also intrdocues the standard notation used for sequences.

1 Sequences and their notation

sequecnes can be notated in shorthand with some special notation, let some sequences be defined as:

$$a_n = \left(\frac{1}{n}\right) \tag{1}$$

$$b_n = \left(n^2\right)$$

The terms in the sequence would be :

$$(a_n)_{n=1}^{\infty} = (1, \quad 1/2, \quad 1/3, \quad 1/4, \quad 1/5, \dots)$$

$$(b_n)_{n=1}^{\infty} = (1, \quad 4, \quad 9, \quad 16, \quad 25, \dots)$$

2 The limit of a sequence and definition of convergence

Sequences which have an infinite number of terms will either diverge or converge. If the sequence diverges then it does not have a limit and may go to infinity or fluctuate over some numbers throughout the terms in the sequence. When the sequence is convergent it will converge to a limit, e.g the sequence (1) will have $\lim_{n\to\infty}(a_n) = a$ as a limit. The formal definition of convergence is[2, P.4] :

$$\forall \varepsilon > 0, \varepsilon \in \mathbb{R} \quad \exists N(\varepsilon) \in \mathbb{N} : \forall n \geq N(\varepsilon) : |a - a_n| < \varepsilon$$

This definition is used to decide whether a sequence has a limit and if it is convergent. if the statemnt does not hold true for a sequnce e,g (b_n) then it doesn't have a limit and is therefore diveregent.

2.1 example of a convergence

to see how sequence the (a_n) is convergent we will use the definition and a graph to show things more clearly. If we choose $N(\epsilon) > 1/\epsilon$ and note that $(a_n) = (1/n)$ then we see that the inequality holds true for all $N(\epsilon) > 1/\epsilon$:

$$1/n \leq 1/N(\epsilon) < \frac{1}{(1/\epsilon)} = \epsilon$$

so for whatever value of ϵ we choose we can find a natural number $N(\epsilon)$, such that the inequality remains true for all of the succeding $_nth$ terms in the sequence. This means that $\lim_{n\to+\infty}(a_n) = a = 0$

$$|a - a_n| = |0 - a_n| = |a_n| < \epsilon$$

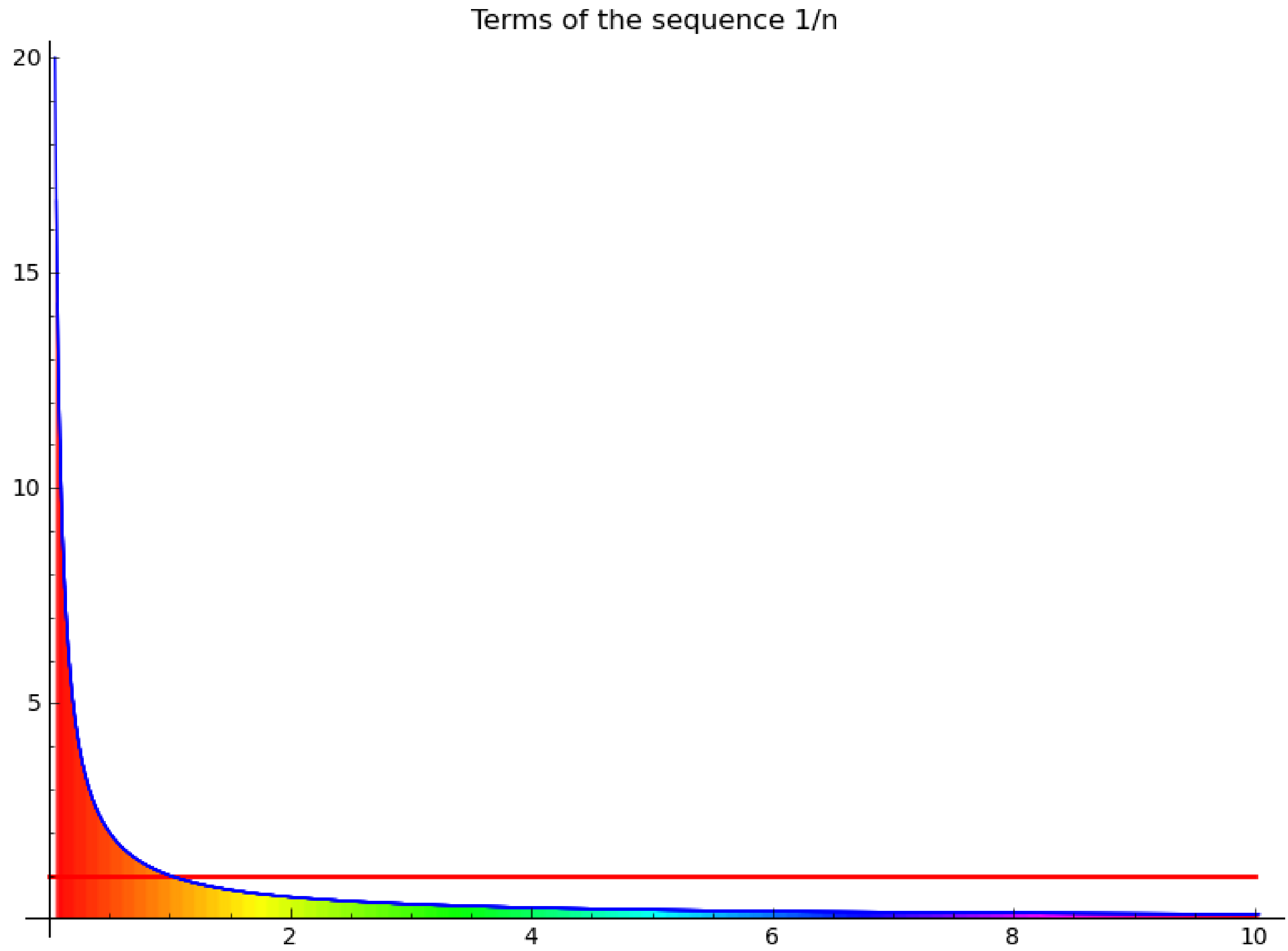

Fig 1. This is a graph showing convergence of the sequence (a_n), the red line is $\epsilon = 1$.

The graph shows that all the terms of the sequence will be smaller than ϵ after some $N(\epsilon)$, also as n heads towards $+\infty$, (1/n) heads towards 0, so:

$$\lim_{n \to +\infty} \left(\frac{1}{n} \right) = 0$$

2.2 Example of a divergence

With this exaple it is more clear to see how this sries is divergent, the graph shows that the terms in the sequence do not approach a limit they increase without bound and tend to $+\infty$. To show that the sequence $b_n = \left(n^2\right)$ is not convergent we shall first assume it is convergent and then show a contradicition. If we assume that (b_n) is convergent then $\forall \epsilon > 0, \exists N(\epsilon) : \forall n \geq N(\epsilon) : |b - b_n| < \epsilon$.
If we choose $N(\epsilon) < \sqrt{\epsilon} \Rightarrow N(\epsilon) \leq n \leq \sqrt{\epsilon} \Rightarrow (N(\epsilon))^2 \leq n^2 \leq \epsilon$ so the strict inequality $n < \epsilon$ is not true $\forall n$. Furthermore since the definition has to be valid $\forall n \geq N(\epsilon) \implies \forall \epsilon > 0, \exists n : n^2 \geq \epsilon$, this is because the (infinite) set of natural numbers $\mathbb{N}$ is not bounded from above, [1] so there will always be a bigger $N(\epsilon)$ we can select so the inequality becomes false for all $n \geq N(\epsilon)$. Now we have shown the sequence (b_n) is unbounded and tends to $+\infty$. We may write this as $\lim_{n \to +\infty}(b_n) = +\infty$, which means it *increases without bound* **and is not convergent and doesn't have a limit.**

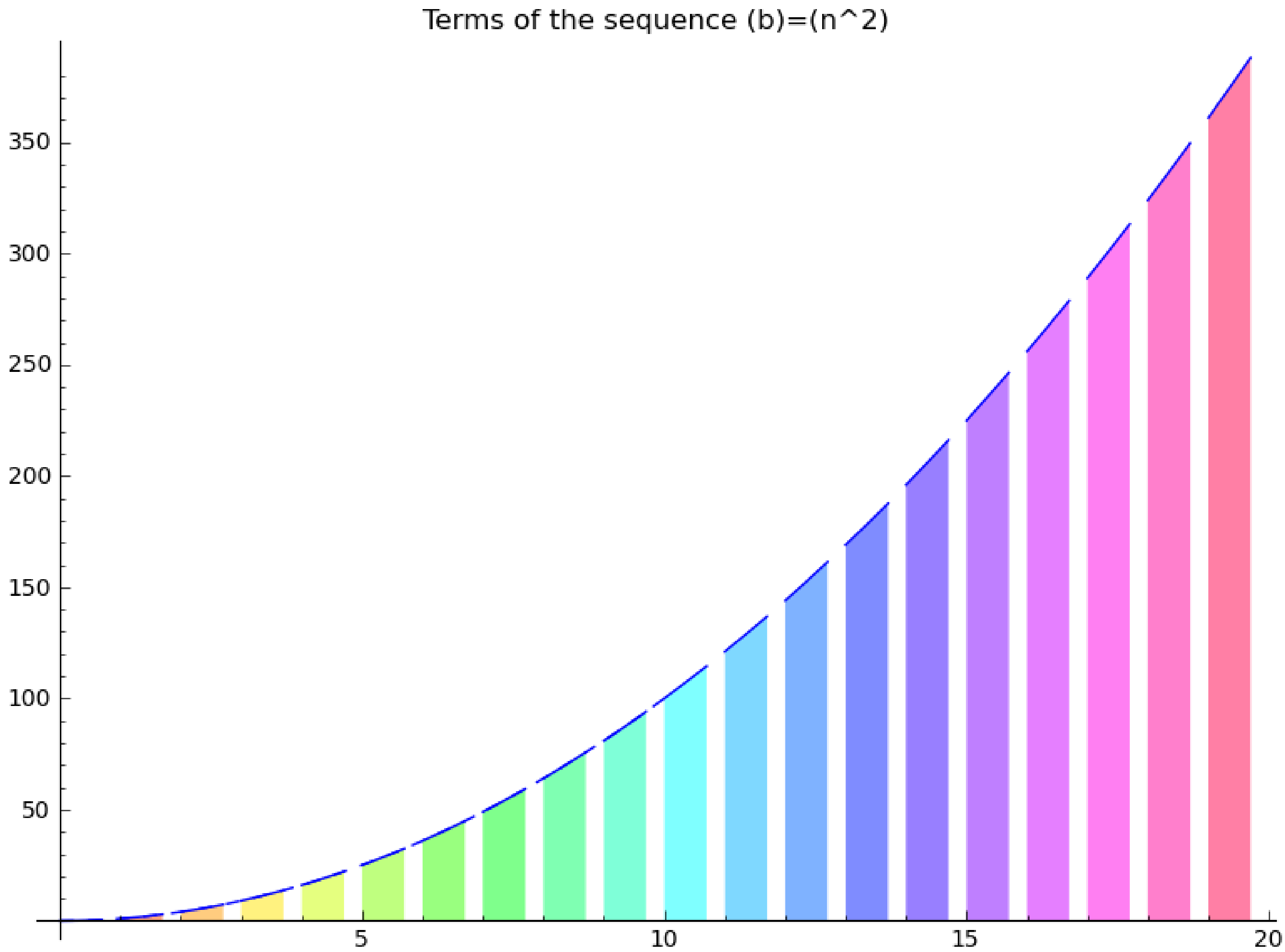

Fig 2. This is a graph showing the divergence of the sequence (b_n)

The graph shows that as n grows larger and tends to $+\infty$, (n^2) also tends to $+\infty$. This means that:

$$\lim_{n\to+\infty}(n^2) = +\infty$$

2.3 Conclusion

Fom this report I have learnt how to use latex and sage toghether to explain the definition of convergence by using the mathematical notation in Latex. I have learnt to plot graphs in sage and import them to Latex to show a sequence converging, it was interesting to learn how graphs and images can be placed in latex and also how to cite many references clearly in the bibliography.

References

[1] Wikipedia , Anonymous. Upper and lower bounds. `http://en.wikipedia.org/wiki/Upper_and_lower_bounds/`.

[2] J.B. Reade. *An introduction to mathematical analysis*. Oxford science publications. Clarendon Press, 1986.