THE LOEWNER EQUATION AND THE DERIVATIVE OF ITS SOLUTION

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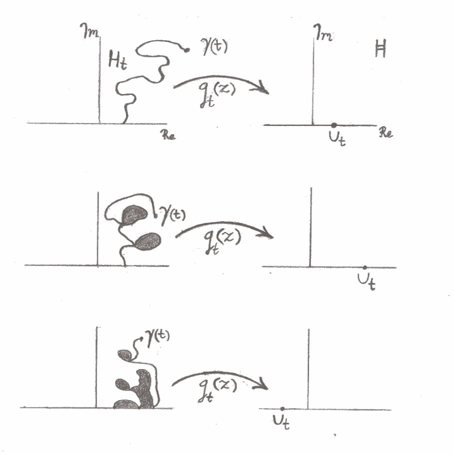
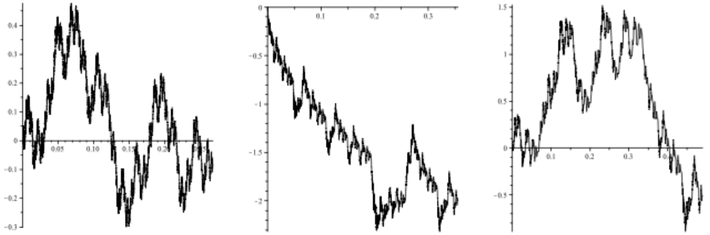
Master's Thesis presentation, KTH & SU

Charles Loewner and the introduction of the Loewner Equation.



Illustrations

Some examples of driving functions U_t



Source: ”Spacefilling Curves and Phases of the Loewner Equation”, J.Lind, S.Rohde

And the sets they generate

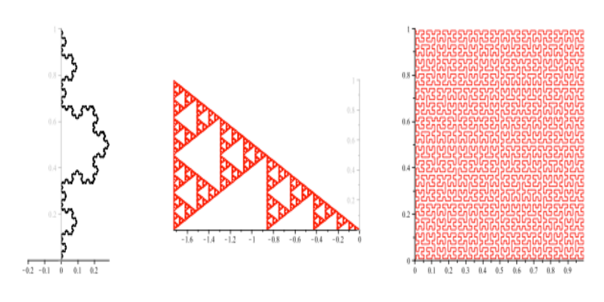


Figure: van Koch curve, the half-Sierpinski gasket, and the Hilbert space-filling curve

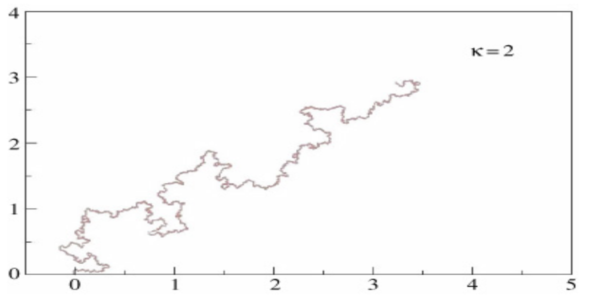
Source: "Spacefilling Curves and Phases of the Loewner Equation", J.Lind, S.Rohde

the SLE-curve

Let $U_t = \sqrt{\kappa} B_t$, where $\kappa \in \mathbb{R}$ and B_t is a standard Brownian motion. Then, with probability one, the set H_t is generated by a curve, i.e $H_t = \mathbb{H} \backslash \gamma(0, t]$ for some continuous curve γ. This γ is called an SLE-curve. Two interesting facts about the SLE-curve:

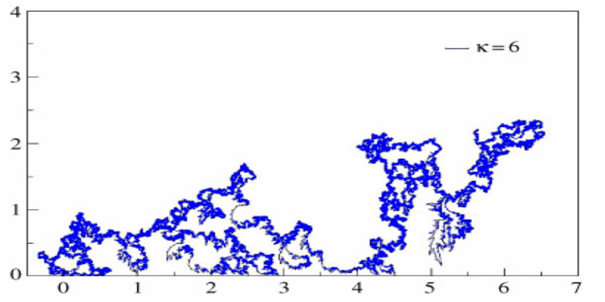
- The curve spirals at every point
- The trace is extremely sensitive to the value of κ. In fact
  - For $0 \leq \kappa \leq 4$ the trace γ is simple with probability one.
  - For $4 \leq \kappa < 8$ the trace γ intersects itself and every point is contained in a loop but the curve is not space-filling (with probability 1).
  - For $\kappa \geq 8$ the trace γ is space-filling (with probability 1).

$\kappa = 2$, yielding a simple trace:



Source: http://iopscience.iop.org

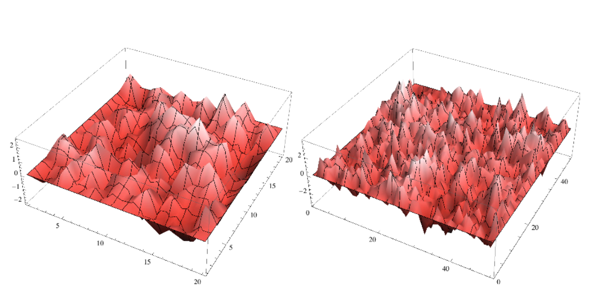
$\kappa = 6$, yielding a non-simple trace:



Source: http://iopscience.iop.org

Discrete Gaussian free fields

Rougher grid left, finer grid right



Source: ”Finding SLE paths in the Gaussian free field ”, S.L. Watson

Purpose of Third part

Aim: Investigate upper bounds of the quantity $|\arg g_t^{-1\prime}(z)|$ as z approaches the boundary, for driving functions of Hölder-1/2 continuous driving functions
(Recall Hölder-1/2 continuity: $|U_{t+s} - U_t| \leq \sigma\sqrt{s}$ for some $\sigma \in \mathbb{R}^+$)

Apriori expectations

- $\boldsymbol{\sigma} < 2\sqrt{2}$: Nontrivial bound for argument should exist and this should be possible to prove with methods from RTZ
- $2\sqrt{2} \leq \boldsymbol{\sigma} < 4$. Non-trivial bound should exist for the argument since it does exist for the absolute value for $\sigma < 4$; but this cannot be proved with methods outlined in RTZ

An illuminating example

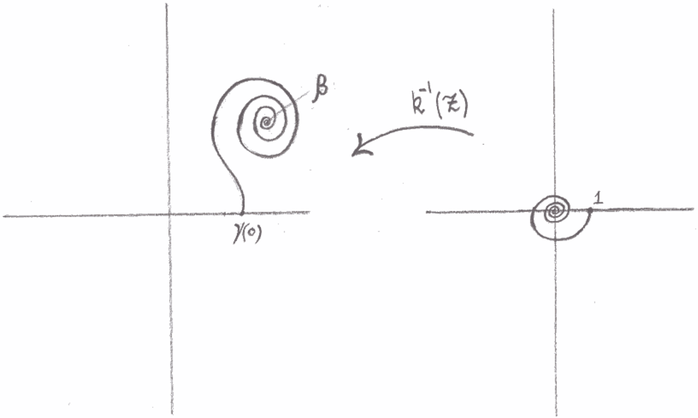
In the article "Collisions and spirals of Loewner traces" by Lind, Marshall and Rohde, the Loewner trace and conformal map of the driving function $U_t = \sigma\sqrt{1-t}$, $0 < \sigma < 4$ are calculated. In fact here the following is stated: Given $0 < \sigma < 4$, set $\theta := -\sin^{-1}(\sigma/4)$, $\beta = 2ie^{i\theta}$, and set

$$k(z) = \frac{(z-\beta)(z-\bar{\beta})^{e^{2i\theta}}}{(\sigma-\beta)(\sigma-\bar{\beta})^{e^{2i\theta}}}$$
$$g_t(z) = (1-t)^{1/2} k^{-1}((1-t)^{-\cos\theta e^{i\theta}} k(z))$$

Then k is a conformal map of $\mathbb{H}$ onto $\mathbb{C}\backslash G$ where $G := \{e^{te^{i\theta}}; t \geq 0\}$, is a logarithmic spiral in $\mathbb{C}$ beginning at 1 and tending to ∞, and where g_t satisfies the Loewner equation

$$\dot{g}_t = \frac{2}{g_t - \sigma\sqrt{1-t}} \quad g_0 \equiv z$$

The trace $\gamma = k^{-1}(\{e^{-te^{i\theta}}; t > 0\})$ is a curve in $\mathbb{H}$ beginning at $\sigma \in \mathbb{R}$ spiraling around $\beta \in \mathbb{H}$


β
k⁻¹(z)
γ(0)
1

Results

- $\sigma < \sqrt{2}$: A non-trivial bound is obtained in line with expectation.
- $\sqrt{2} \leq \sigma < 2\sqrt{2}$: We show the methods of RTZ are insufficient for obtaining a non-trivial bound in this interval. However, we still suspect such a nontrivial bound to exist
- $2\sqrt{2} \leq \sigma < 4$: We have shown no non-trivial bound can exist in this interval

Questions?