Stochastic Process Notes

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1 Martingales

1.1 Simple Random Walks

A SRW is a (martingale/submartingale/supermartingale) if $p(= / > / <)\frac{1}{2}$

1.2 multi-step propery

Just use tower property multiple times

1.3 convex/concave, integrable functions of martingales

These are submartingales/supermartingales; just use Jensen's conditional

1.4 increasing ϕ and M submartingale

Then $\phi(M_n)$ is a submartingale; order preserved

2 Martingale Transforms

2.1 Martingale transforms of predictable processes wrt martingales are martingales

Write $E[I_{n+1}|\mathcal{F}_n] = I_n + H_{n+1}E[X_{n+1} - X_n|\{_n]$ and use properties of conditional expectation and filtrations.

2.2 Martingale transforms of predictable, bdd, increasing processes wrt submartingales are submartingales

Do same as previous result

3 Stopping Times

3.1 Form of stopping time for natural filtrations

$\mathbb{1}_{\{}T = n\}$ can be written as $g_n(X_0, \dots, X_n)$ for some measurable g_n. To prove, just use definition of $\sigma(X_0, \dots, X_n)$ measurability

3.2 Predictability of $H_n = \mathbb{1}_{\{n \leq T\}}$

Write $\mathbb{1}_{\{n \leq T\}} = 1 - \mathbb{1}_{\{T<n\}} = 1 - \mathbb{1}_{\{T \leq n-1\}}$

3.3 $X_{n \wedge T}$ is a super/sub/martingale

True since it can be written as $(\mathbb{1}_{\{n \leq T\}} \cdot X_n) + X_0$

4 Optional Stopping Theorem with bdd stopping times (unconditional version)

The result says that if $S \leq T < k < \infty$ and M_n is a sub/super/martingale then $E[M_S](\leq / \geq / =)E[M_T]$

4.1 $M_{n \wedge T} - M_{n \wedge S}$ is a submartigale (follow same reasoning for super and martingale)

$M_{n \wedge T} - M_{n \wedge S}$ can be written as $(\mathbb{1}_{\{n \leq T\|} \cdot$ which, in turn, is a submartingale

4.2 $E[M_{n \wedge T} - M_{n \wedge S}] \geq 0$

True because of the previous result, but I don't know why. Let $n = k$ to finish result

5 $\mathcal{F}_S$ for S a stopping time

General definition is $\mathcal{F}_S = \{A \in \mathcal{F} | A \cap \{S = n\} \in \mathcal{F}_n \text{ for all n}\}$. It is straightforward to show that this is a σ-algebra

5.1 Definition reduction to $\sigma(X_0, X_{1 \wedge S}, \ldots, X_{n \wedge S}, \ldots)$ in case of natural filtration

I do not know why this is true

5.2 $L = S\mathbb{1}_A + T\mathbb{1}_{A^C}$ is a stopping time if $A \in \mathcal{F}_S$ and $S \leq T$

Homework problem

5.3 $E[M_S] \leq E[M_L] \leq E[M_T]$ when $S \leq T < k < \infty$

This is just the unconditional version of the OS Theorem

5.4 $M_L = M_S \mathbb{1}_A + M_T \mathbb{1}_{A^C}$

I don't know why this is true

5.5 Conditional OS Theorem: $M_S \leq E[M_T | \mathcal{F}_S]$

Substitute 5.4 into 5.3 to get $E[M_S \mathbb{1}_A] \leq E[M_T \mathbb{1}_A]$ then use definition of conditional expectation.

5.6 $M_S \mathbb{1}_{\{S<\infty\}}$

Just write this as $\sum_{i=0}^{\infty} M_i \mathbb{1}_{\{S=i\}}$ then examine preimage of rays

6 Up-crossings and Up-crossing inequality

Setup: let $a < b$ and define
$T_0 = 0$
$T_{2k+1} = \inf\{n \geq T_{2k} | M_n \leq a\}$
$T_{2k+2} = \inf\{n \geq T_{2+1k} | M_n \geq b\}$
$U(a, b, n) = \max\{K | T_{2k} \leq n\}$
$U(a, b) = \lim U(a, b, n)$
$H_n = \sum_{k=0}^{\infty} \mathbb{1}_{\{T_{2k+1}<n\leq T_{2k+2}\}}$

6.1 Up-crossing inequality: $(H \cdot M) \geq (b - a)U(a, b, n) +$ possible final loss

Just think about it for a minute

6.2 Up-crossing Theorem: $E[U(a, b, n)] \leq \frac{E[M_n^+]+|a|}{b-a}$ if M_n is a submartingale

Note that since $H_n \leq 1$, $((1 - H) \cdot M)$ is a nonnegative submartingale. Hence $E[(H \cdot M)_n] \leq E[M_n - M_0]$. Now replace M_n with $N_n = (M_n - a)^+$ which is a submartingale (x^+ is positive, convex, increasing). Use the estimate: $E[N_n - N_0] \geq E[(H \cdot N)_n] \geq E[(b - a)U(a, b, n)]$ to get the inequality.

7 Martingale Convergence Theorem

Assume M_n is a submartingale and $\sup_n M_n^+ < \infty$ (M_n^+ is bdd in L^1).
Then there is some $M_\infty \in L^1(\mathcal{F}_\infty)$ s.t. $M_n \to M_\infty$ a.s. where $\mathcal{F}_\infty = \bigvee_n \mathcal{F}_n$

7.1 If M_n is a submartingale then L^1 boundedness of M_n boundedness is equivalent to L^1 boundedness of M_n^+

Observe that $E[M_n] = E[M_n^+] - E[M_n^-]$ where the first two expectations are increasing in n (since M_n and M_n^+ are submartingales) and the third is positive. If M_n is bounded in L^1, then M_n^+ is bounded in L^1 since M_n^+ is dominated by $|M_n|$. Now if M_n^+ is bounded in L^1 then $E[|M_n|]$ is bounded since M_n^+ dominates M_n. It follows that M_n^- is bounded in L^1 which, in turn, makes M_n bounded in L^1.

7.2 $P(U(a, b) < \infty) = 1$

By the Up-crossing theorem and the fact that M_n^+ is bounded in L^1, $E[U(a, b)] \leq \sup \frac{E[M_n^+ + |a|]}{b-a} < \infty$. The result holds as a consequence of boundedness of the integral.

7.3 M_n converges to some $M_\infty \in \mathcal{F}_\infty$ a.s.

By definition of limsup and liminf,
$\{\underline{\lim} M_n < \overline{\lim} M_n\} = \cup_{a<b|a,b\in\mathbb{Q}}\{U(a,b)=\infty\}$. Hence,
$P(\underline{\lim} M_n = \overline{\lim} M_n) = 1$. Furthermore, $M_\infty \in \mathcal{F}_\infty$ since $M_n \in \mathcal{F}_n \subset \mathcal{F}_\infty$. Fatou can be used to get L^1 boundedness: $E[|M_\infty|] \le \underline{\lim} E[|M_n|] < \infty$

7.4 First hitting time at a point for a simple random walk is finite a.s.

Given $a > 0$, define $T = \inf\{n \ge 0 | X_n = a\}$. Then $X^+_{T\wedge n} \le a \Rightarrow X^+_{T\wedge n}$ is L^1 bounded. Since the random walk, X_n is a martingale (and hence a submartingale), so is $X^+_{T\wedge n}$. By the martingale convergence theorem, $X_{n\wedge T}$ converges almost surely to something that is a.s. finite. As a result,
$P(X_{n\wedge T}$ is eventually constant$) = 1$ and hence $P(T < \infty) = 1$.

7.5 Counterexample showing deficiency in O.S. Theorem with unbounded times

Note that, in the simple random walk, $0 = X_0 \ne E[X_T] = a$. O.S. theorem does not apply since T is unbounded. Additionally note that, though $X_{n\wedge T} \to 1$ a.s., convergence in L^1 does not occur. Uniform continuity is the missing condition.

8 Uniform Integrability (UI)

In these notes, uniform integrability will be with regard to a collection of random variables, χ unless stated otherwise

8.1 If χ is dominated by some $y \in L^1$ then χ is UI

Use domination and DCT, to get $E[|X|\mathbb{1}_{\{|X|\ge M\}}] \le E[|Y|\mathbb{1}_{\{|Y|\ge M\}}] \to 0$ as $m \nearrow \infty$ for all $X \in \chi$

8.2 If χ is countable then domination is equivalent to sup$|X| \in L^1$

This is clearly true for a countable χ. The real question is: why is it not generally true for uncountable χ

8.3 If $X_n \to X$ in probability then the subsequent implications are true (giving TFAE)

8.4 X_n is UI $\Rightarrow X_n \to X$ in L^1. Additionally, X_n and $X \in L^1$

First define $\phi_m(x) = -M\mathbb{1}_{\{x\le -M\}} + x\mathbb{1}_{\{x\in(-M,M)\}} + M\mathbb{1}_{\{x\ge -M\}}$ and observe that $|x - \phi_M(x)| = (|X| - M)^+\mathbb{1}_{\{|X|\ge M\}} \le |X|\mathbb{1}_{\{|X|\ge M\}}$. Also note that UI of X_n implies that $X_n \subset L^1$. Now, just use the estimate $E[|X_n - X|] \le$
$E[|X_n - \phi_M(X_n)|] + E[|\phi_M(X_n) - \phi_M(X)|] +$
$E[|\phi_M(X) - X|] \le E[|X_n|\mathbb{1}_{\{|X_n|\ge M\}}] + E[|\phi_M(X_n) - \phi_M(X)|] + E[|X|\mathbb{1}_{\{|X|\ge M\}}]$
obtained from the triangular inequality and the above observation. The first

term in final part of the estimate vanishes due to UI, the third vanishes due to the fact that $X \in L^1$, and the second vanishes due to.

8.5 $X_n \to X$ in L^1 where X_n and $X \in L^1 \Rightarrow E[|X_n|] \to E[|X|]$

This is just triangular inequality

8.6 $E[|X_n|] \to E[|X|] \Rightarrow X_n$ is UI